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HOLLOW MODULE

A research submitted to the Council of the College of Education/ Maysan University, Department of Mathematics, which is part of the requirements for obtaining a bachelor's degree in mathematics.

By

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بِسْمِ اللهِ الرَّحْمنِ الرَّحِيم

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَات

صدق الله العلي العظيم

[سورة المجادلة : 11]

الإهداء

"وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ"
الحمد لله عند البدء وعند الختام
من قال انا لها نالها

لقد كان طريقاً مليئاً بالإخفاقات والنجاحات فخورين بكفاحنا لتحقيق حلمنا لحظه لطالما انتظرتها وحلمت بها في حكاية اكتملت فصولها
الى من علمني العطاء بدون انتظار الى من احمل اسمه بكل افتخار إلى والدي العزيز
والى حبيبتي قرة عيني الى من كانت دعواتها الصادقة سر نجاحي امي الغالية
والى كل افراد عائلتي بدون استثناء
والى كل الأساتذة الأفاضل الذين قدموا لنا يد المساعدة
الى كل هؤلاء اهدي هذا العمل وفقني وإياكم بالخير

الشكر والامتنان

أتقدم بخالص الشكر والتقدير إلى الله تعالى الذي وفقني لإتمام هذا البحث، ثم إلى [الأستاذ عبدالكريم]، أستاذي الفاضل، على توجيهاته القيمة ودعمه المستمر طوال فترة إعداد البحث، ولا أنسى أن أشكر عائلتي وأصدقائي الذين كانوا مصدراً دائماً للدعم والتشجيع.

Supervisor approval

I certify that this research (Hollow Module) submitted by the student (Tabark Ali Mohammed) , took place under my supervision at the University of Maysan / College of Education / Department of Mathematics. It is part of the requirements for obtaining a bachelor's degree in the College of Education / Department of Mathematics.

Supervisor

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TABLE OF CONTENTS

ABSRACT

Let R be any ring with identity and let M be a unitary left R–module. This research studies two types of concepts . The first is small submodules and the second hollow modules with some examples and properties.

INRODUCTION

Throughout, all rings are associative with identity, and modules are rings unitary. In this work, we will study the concepts of the small submodules and the hollow modules This research has two chapters: In chapter one, we recall some properties about the module. In chapter two, there are two sections. In section one, study the small submodules with examples and properties. Section two studies the hollow modules with examples and properties.

CHAPTER ONE

BASIC CONCEPT OF MODULES

Chapter One : Basic Concept Of Modules.

In this chapter we will recall the definition of modules with some examples and properties.

Definition (1. 1)[1]: A non- empty set (G) on which binary operation (+) is defined is said to from group with respect to this operation provided for arbitrary a, b, c

- **The following properties hold :**

1. (a*b) *c =a*(b*c)
2. There exists 0 ∈G s.t a*0= a for all a ∈G
3. For each a ∈ G there exists a – 1 ∈ G s.t a*a – 1 =0

Example (1.2) : (Z, +) is group.

Solution :

a) Z is closed under (+) → a+b ∈ Z ∀ a, b ∈ Z.
b) (+) is associative on Z → (a+b)+c = a +(b +c),∀a, b, c ∈ Z.
c) 0 is the identity element with add since ∀ a ∈ Z → a+0 = 0+a =a.
d) ∀ a ∈ Z, ∃ -a ∈Z s.t a+ (-a) =0.

Definition (1.3)[1]: A group (G, *) be called commutative group or

(abelian group) if a*b = b*a, ∀ a, b ∈ Z.

Example (1.4) : show that (Q – {0}, •) is commutative group.

Solution :

1) Q-{0} is closed under (•), since ∀ a, b ∈ Q-{0} → a+b ∈ Q-{0}.
2) (•) is assoc. on Q-{0} , since (a. b). c = a. (b. c) .

3) 1 is the identity element with multiple of Q-{0},∀ a∈ Q-{0} →a.1=1.a =a
4) ∀ a∈ Q-{0}, ∃ a– 1 ∈ Q-{0} s.t a. a – 1 =1.
5) ∀a, b ∈ Q-{0} → a. b= b. a → (•) is comm. On Q-{0}.

∴ (Q-{0} , •) is comm. group.

Definition (1.5)[1]: let R be non– empty set and let (+, •) be two binary operations on R then

❖ **(R, +,•) is a ring if :-**

1. (R, +) is an abelian group that is :
   - + is closed on R.
   - + is associative.
   - ∀ a ∈ R, ∃ 0 ∈ R s.t a + 0 = a .
   - ∀ a ∈ R, ∃ - a ∈ R s.t a + (-a) = 0.
   - + is commutative.
2. (a. b). c = a. (b. c) .
3. a. (b+c) = a. b+a. c .

Example (1.6): show that (Z, +, •) is a ring.

Solution :

1) (Z, +) is an abelian group :

   a. Closed, ∀a, b ∈Z s.t a+b ∈ Z.
   b. Assoc. ∀a, b, c, ∈Z s.t (a +b) +c = a+(b+c).
   c. ∀a ∈Z, ∃ e∈Z s.t a+e =a .
   d. ∀a ∈Z, ∃ - a ∈Z s.t a+(-a) = e .
   e. Comm., ∀a, b ∈Z s.t a+b =b+a .
2) (Z, •) is semi group :
   a) Closed, ∀a, b ∈Z s.t a. b ∈Z.
   b) Assoc. , ∀ a, b, c ∈Z s.t (a. b). c = a. (b. c) .

3) Distributive
   $\forall a, b, c \in Z$ s.t $a.(b+c) = a.b + a.c$

<u>Definition (1.7)[1]:</u> let R be a ring with Identity an abelian group (M, +) is called a left R-module if there exists a mapping $\propto: R\times M \to M$ s.t $\propto(r, m) = r.m$, $\forall\, r \in R, m \in M$.

Set is fying the following conditio :

1. $\propto(r, m_1 + m_2) = \propto(r, m_1) + \propto(r, m_2)$.
2. $\propto(r_1 + r_2, m) = \propto(r_1, m) + (r_2, m)$.
3. $\propto(r_1.r_2, m) = \propto(r_1), \propto(r_2, m)$.
4. In addition $1.m = m$.

Example (1.8) : Every additive abelian group is Z–module.

Solution : let (M, +) be an abelian group define a mapping $\emptyset: Z\times M \to M$ s.t

$$\emptyset(n, a) = n.a \text{ where } n.a = \begin{cases} a + a + \ldots\ldots + a (n\text{– times}) \text{ if } n > 0 \\ (-a) + (-a) + \ldots. + (-a)\ (n\text{– times}) \text{ if } n < 0 \end{cases}$$

Now, we satisfying the conditions :-

1. $\emptyset(n, a1 + a2) = n(a_1 + a_2)$
   $= (a_1 + a_2) + \ldots\ldots + (a_1 + a_2)$ (n–times)
   $= \underbrace{(a_1 + a_1 + \ldots.. + a_1)}_{n\text{- times}} + \underbrace{(a_2 + a_2 + \ldots. + a_2)}_{n\text{- times}}$
   $= n.a_1 + n.a_2$
   $= \emptyset(n, a_1) + \emptyset(n, a_2)$
2. $\emptyset(n_1 + n_2, a) = (n_1 + n_2).a = n_1.a + n_2.a$
3. $\emptyset(n_1.n_2, a) = (n_1.n_2).a = n_1.(n_2.a)$

4. If 1 is the identity element of ring Z , then 1.a = a , ∀ a ∈ M

Definition (1.9)[1]: Let M be a right R-module and let ∅ ≠ N ≤ M is called a submodule of a module M

iff :-

a. a + b ∈ N , ∀ a, b ∈ N.
b. r. a ∈ N , ∀ r ∈ R and a ∈ A

Example (1.10) : (Z, +) is Submodule of Module (Q, +) over a ring (Z, +) Since Q ≠ Z ⊆ Q .

Solution :

a. a + b ∈ Z, ∀ a, b ∈ Z.
b. r. a ∈ Z, ∀ r ∈ R, ∀ a ∈ Z.

Definition (1.11)[1]: A Submodule N of left R–module M is said to be a direct summand of M if there is a Submodule of M s.t M = N ⊕K. In other word there is a Submodule K of M s.t M = N + K and N ∩ K = 0 .

Example (1.12) : let M = Z_6 as a left Z– module , find all direct summand of M.

Proof : M, 0, N_1 = < 2 > = { $\bar{0}$,$\bar{2}$, $\bar{4}$ } and N_2 = < 3 > = { $\bar{0}$, $\bar{3}$ } are all directs summand of M .

Definition (1.13)[1]: let M be an R–module called semi simple module if every Submodule of M is a direct summand of M.

Definition (1.14)[1]: suppose that A, B and C are Submodule of M over a ring R and suppose that A ≤ B, show that (B ∩ C) + A = B ∩ (C + A).

Definition (1.15)[1]: let G and H be group. A homo. F : G →H is function

$f : G \rightarrow H$ s.t for all $g_1, g_2 \in G$, $f(g_1, g_2) = f(g_1) f(g_2)$.

Definition (1.16)[1]: If $f : A \longrightarrow B$ is a homomorphism and every element of B is the image of some element in A , then *f* is an Epimorphisms.

Definition (1.17)[1] : A submodule N of a module M called a proper submodule if $N \neq M$.

☒ **in simple terms :**

- $N \subseteq M$
- N is closed under addition and scalar multiplication (i.e., it's a submodule)
- But not equal to the whole module.

CHAPTER TWO

SMALL SUBMODULES AND HOLLOW MODULES

Chapter Two : Small Submodule And Hollow Modules.

In this chapter, we will recall the concepts of the small submodules and hollow modules with some properties.

1. SMALL SUBMODULE

In this section, we recall the small submodule with some properties.

Definition (2.1.1)[2] :

Let M be an R–module and S be a submodule of M , S is said to be small submodule of M (denoted N M) , if for any submodule N of M such that M = S + N, we have N = M .

Examples and Remarks (2.1.2) :

1) For any R–module M , $\{ \overline{0} \}$ is a small submodule of M but is not small in M . Since the only case that we have is 0 + M = M

2) The submodule $\{ \overline{0} , \overline{2} \}$ of the $\mathbb{Z}_4$ as $\mathbb{Z}$–module is small of $\mathbb{Z}_4$ since the only case that we have is $\{ \overline{0} , \overline{2} \} + \mathbb{Z}_4 = \mathbb{Z}_4$.

3) The submodules $\{ \overline{0} , \overline{2} , \overline{4} \}$ of the $\mathbb{Z}_6$ as $\mathbb{Z}$–module is not small of $\mathbb{Z}_6$

since $\{ \overline{0} , \overline{2} , \overline{4} \} + \{ \overline{0} , \overline{3} \} = \mathbb{Z}_6$ but $\{ \overline{0} , \overline{3} \} \neq \mathbb{Z}_6$.

4) In $\mathbb{Z}$ as a $\mathbb{Z}$-module, $2\mathbb{Z}$ is not a small submodule since $2\mathbb{Z}+3\mathbb{Z}=\mathbb{Z}$ but $3\mathbb{Z}\neq\mathbb{Z}$.

The following proposition gives some properties of small submodules see [3] .

Proposition (2.1.3):

Let K_1 and K_2 be a submodules of R–module M $K_1 \ll M$ and $K_2 \ll M$, then $K_1 + K_2 \ll M$.

Proof :

Let $K_1 \ll M$ and $K_2 \ll M$ to show $K_1 + K_2 \ll M$, let $K_1 + K_2 + U = M$,where U submodule of M to prove M = U since $K_1 \ll M$,

then $K_2 + U = M$ since and since $K_2 \ll M$, then M = U and $K_1 + K_2 \ll M$.

The following proposition shows that the image of a small submodule is also small.

Proposition (2.1.4) : Let $f : M \rightarrow M'$ be an R-epimorphisms and $A \ll M$. then $f(A) \ll M'$.

Proof :

Let $f(A) + B = M'$ where B is a submodule of M'. So $f^{-1}(B)$ is a submodule of M' and $A + f^{-1}(B) \leq M$ …(1)

Let $m \in M$. Hence $f(m) \in M' = f(A) + B$. So there exists $a \in A$ such that $f(m) = f(a) + b$ for some $b \in B$ and $b = f(m-a)$.

Hence $m-a \in f^{-1}(B)$. Since $a \in A$. So $m = m-a+a \in A + f^{-1}(B)$.

Thus $M \leq A + f^{-1}(B)$ …(2).

From (1) & (2) have that $A + f^{-1}(B) = M$ since $A \ll M$, so $f^{-1}(B) = M$ and $ff^{-1}(B) = f(M)$ since f is onto therefore $B = M'$ and $f(A) \ll M'$.

Proposition (2.1.5) : Let M be an R-module M and K_1 and K_2 be Submodule of M with $K_1 \leq K_2 \leq M$ if $K_2 \ll M$ then $K_1 \ll M$.

Proof : Let $K_1 + U = M$ such that U Submodule of M since $K_1 \leq K_2$ we have $M = K_1 + U \leq K_2 + U$, hence $M \leq K_2 + U \leq M$, then $M = K_2 + U$ and since $K_2 \ll M$ therefore $M = U$ and $K_1 \ll M$.

Proposition (2.1.6): Let M be an R-module and N, K, L are Submodule of M with $N \leq L \leq M$, if $K \ll L$ then $N \ll M$.

Proof :

suppose that $N + C = M$, to prove $M = C$, since $L \leq M$,

hence $L = M \cap L = (N + C) \cap L = N + (C \cap L)$

since $N \leq K$, $L = K + (C \cap L)$, since $K \ll L$ then $L = C \cap L$ and $L \leq C$ but $N \leq L$, hence $N \leq C$ then $N + C = M$ then $C = M$ and hence $N \ll M$.

2. HOLLOW MODULE

In this section, we recall the hollow modules with some properties.

Definition (2.2.1) [4] : An R-module M is called hollow modules if every proper submodule of M is small submodule of M

Examples (2.2.2) :

1. Z_4 as Z – module is a hollow module.
2. Z_6 as Z – module is not hollow module.
3. If M is semi simple, then M is not hollow module.

Now , we give some properties of hollow modules. See[5].

Proposition (2.2.3) : Epimorphisms image of hollow module is hollow.

Proof : Let M be hollow module and $f : M \to M'$ an epimorphisms with M' is module.

Suppose N is proper submodule of M' with $N' + K' = M'$ where $K' \leq M'$

Now $f^{-1}(N')$ is proper submodule of M since otherwise $f^{-1}(N') = M$ and hence $f(f^{-1}(N')) = f(M) = M'$ implies that $N' = M'$,

thus $f^{-1}(N')$ is proper submodule of M and therefore $f^{-1}(N') \ll M$ and hence $f(f^{-1}(N')) \ll f(M)$ that is $N' \ll M'$.

Proposition (2.2.4) : Let K be small submodule of module M if M / K is hollow module then M is hollow.

Proof : Suppose that M / K is hollow module with $K \ll M$.

Let N be proper submodule of M with $M = N + L$,

where $L \leq M$ then $M / K = (N + L) / K$

implies that $M / K = [(N + L) / K] + [(L + K) / K]$ since $(N + L) / K$ is proper submodule of M / K then $(N + L) / K \ll M / K$ and hence

$(L + K) / K = M / K$ therefore $L + K = M$ but $K \ll$ M then $L = M$ that is

M is hollow module.

Proposition (2.2.5) :

Let $M = M_1 \oplus M_2$, M is duo module then M is hollow iff M_1 and M_2 are hollow provided $N \cap M_i \neq M_i$ for all $i = 1 , 2$, $N \subseteq M$.

Proof :

⇒) let M is hollow and $N_1 \oplus N_2 \subseteq M_1 \oplus M_2$ with $N_1 \subseteq M_1$ and $N_2 \subsetneq M_2$ and $N_1 \oplus N_2 \ll M_1 \oplus M_2 = M$ to show M_1 is hollow.

Let $\pi_1 : M_1 \oplus M_2 \longrightarrow M$ the projection map define as follows, $\pi_1(m_1 + m_2) = m_1$ for all $m_1 + m_2 \in M_1 \oplus M_2$, since $N_1 \oplus N_2 \ll M_1 \oplus M_2$. Thus by proposition (2.1.4) $\pi_1(N_1 \oplus N_2) \ll \pi_1(M_1 \oplus M_2)$

then $N_1 \ll M_1$. Thus M_1 is hollow and similarly M_2 is hollow.

⇐) let M_1 and M_2 are hollow to show $M = M_1$

To prove $N_1 \oplus N_2 \ll M_1 \oplus M_2$, since $N_1 \ll M_1 \subseteq M$ and $N_2 \ll M_2 \subseteq M$ then by proposition (2.1.5) , $N_1 \ll M$ and $N_2 \ll M$. By proposition (2.1.3).

$N_1 \oplus N_2 \ll M = M_1 \oplus M_2$.

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