Republic of Iraq
Ministry of Higher Education
And Scientific Research
University of Misan
College of Education
Department of Mathematics

Topological space and Metric space

A Thesis Submitted to the council of college of Education/ University of Misan in Partial Fulfillment of the Requirements for the Degree of Bachelor in mathematics.

By
Murtada Nasser

Supervised by
Dr. Murtadha Ali Shabeeb

2025 A.D **1446 A.H.**

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

((يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ))

(المجادلة: ١١)

Dedication

To those who planted the seeds of ambition within me،

To those who stood by my side every step of the way، to my beloved parents — thank you for your endless support and unconditional love.

To my respected professors in the Department of Mathematics — your guidance lit the path of knowledge for me.

To my friends and colleagues who shared this journey with all its sweet and challenging moments.

I dedicate this humble work to everyone who has had an impact on my academic journey.

Acknowledgment

First and foremost, I thank Allah for granting me the strength, patience, and determination to complete this journey.

I would like to express my deepest gratitude to my supervisors and professors in the Department of Mathematics for their valuable guidance, continuous support, and encouragement throughout my academic years.

Special thanks to my parents and family, whose unwavering love and support have been my greatest motivation.

I am also thankful to my colleagues and friends for their cooperation, shared knowledge, and for making this journey a memorable one.

This accomplishment would not have been possible without each one of you. Thank you from the bottom of my heart.

Introduction

This appendix will be devoted to the introduction of the basic proper ties of metric, topological, and normed spaces. A metric space is a set where a notion of distance (called a metric) between elements of the set is defined. Every metric space is a topological space in a natural manner, and therefore all definitions and theorems about topological spaces also apply to all metric spaces. A normed space is a vector space with a special type of metric and thus is also a metric space. All of these spaces are generalizations of the standard Euclidean space, with an increasing degree of structure as we progress from topological spaces to metrics

Conclusion

This research dealt the Study of metric space and topological space, the similarities and differences between them, and the Characteristics of each. It also dealt with defining types of Sets it topological and metric space. We also touched on the tubes of topological Space, and we concluded through this research that topological space is wider that metric space, as the elements of topological space are sets. It is noted that a metric space can be used to define a topological spacer while the reverse is hot true.

Contents

Chapter one

Chapter one

Topology space

[1.1] Definition: "Topology"(1)

It is one of the branches of mathematics that is Concerted with studying the Structures Components and properties of all different space so that these properties remain similar under continuous shaping operations without tearing of leaving openings. in moving from one to the other and vice Versa as well.

[1-2] Definition: "Topological space" (1)

Let X be a nonempty set and τ be a family of X (i.e, $\subseteq IP(X)$). We say, τ is a topology on X if satisfy the following conditions:

(1) $X, \emptyset \in \tau$
(2) If U,V$\in\tau$, Then U $\cap$ V$\in\tau$

The finite intersection of elements from τ is again ah element of τ.

(3) If $U_\alpha \in \tau$: $\alpha \in A$, theh $U_\alpha \in AU_\alpha \in \tau \forall a \in A$

The arbitrary (finite or infinite) union of elements of τ is again as element of τ. We called a pair (X, τ) topological space.

[1-3] Definition: "open set" (1)

Let (X, τ) be any topological space, then the members of τ are said to be an open Set.

[1-4] Example:

If $X = \{a, b\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}\}$ Then $\tau = \emptyset, X, \{a\}$ and $\{b\}$ are open Subset of T.

[1-5] Definition: "Close Set" (1)

Let (X, T) be atopotogical spare. A subset s of X is said to be closed Set in (X, T) if its complements if$X$, hatmetyX/S, is open in(X, T).

[1-6] Example:

$$\text{if } X = \{\text{a}, b\}, T = \{\phi, X, \{\text{a}\}\}$$

Then $X, \phi, \{b\}$ are closed set since

$$\phi^c = X, X^c = \phi, \{a\}^c = \{b\}$$

We will show the types of topologies

[1-7] Definition: "indiscrete of topology" (1)

For any Set $X \neq \phi, I = \{\phi, X\}$ is topology on$X$. I is called the indiscrete or trivial toparagy on X

[1-8] Example $X = \{a, b, c\}, X = \{\emptyset, \text{x}\}$

Solution: [is topology on X and called indiscerte tololagy since satisfies the following conditions:

$$i, X, \phi \in I$$

ii. $\forall A, B \in I \Rightarrow A \wedge B \in I$
iii. $A_i, i \in \text{A} \rightarrow \ UA_i \in IX_1$ is in indiscrete topology s puce.

[1-9] Definition: "Discrete topology" (1)

Let X be atyhoh-emply Set and let I be the collection of all subset of X. then τ is called the discrete topology on the Set X. the topological space (X, T) is called a discrete space.

[1-10] Example:

$$X = \{x\}\ \mathrm{D} = \{\emptyset, X \cdot \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a \cdot c\}, \{a, d\} \cdot \{b, d\} \cdot \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{\mathrm{a}, \mathrm{c}, d\}, \{b, c, d\}, \{c, d\}\}$$

Solution: D is topology on X and called discrete topology since satisfies the following condition:

$i = X, \phi \in D$
ii- $\forall A, B \in D \Rightarrow A \cap B \in D$

iii $- A_i$,i∈ A → $\mathrm{U}A_i$∈D

XD is discrete topological space.

[1-11] Definition: "usual topology"(1)

Let U be a cdlection of all open interval of real numbers, then U is a topotogy on R it called the usual topology on R.

[1-12] Example:

$$X = R, U = \{\phi, R, (-6,6)\}$$

$\emptyset \cdot R \cdot (-6.6)$ ate opeh sets.
R, $\emptyset. (-\infty, -6)] \cup [6, \infty)$ are closed sets.

[1-13] Definition: "Coffinite topology" (1)

let X be aby hoh-ehply set. A tropology τ on X is catted the filite-closed to potogy or the coffinite topology if the closed subset of X ave X and all finite subsets of X; that is, the opec Sets are © ald all subsets of X which have finite Complements.

[1-14] Example:

$$x = Nf^c = \{\phi, X, N \cdot \{1 \cdots \in\}, N - \{10 \ldots, 99\}\}$$
$$X, \phi, N = \{1 \ldots \in\}, N, \{10 \ldots 99\}\tau = \text{ open}$$
$$\phi, X, \{1 \cdots \in\}, \{10 \ldots 99\}\tau \text{ closed}$$

[1-15] Definition: Compact Space (1)

a space X is called compact if each open cover of X has a finite sub cover for$X$ i.e., X is compact

$\forall \mathrm{c} = \{U_\alpha\}\alpha \in A;\ U_\alpha \in \tau\ \forall\alpha \cap X = U_\alpha \in AU_\tau\alpha$

$\Rightarrow \nexists, \alpha_1 \dots \alpha_\mathrm{n}, X = U_{\mathrm{i}=1}^{n} U\alpha i.$

X is not Compact $\Rightarrow \exists C = \{U\ \alpha\}\ \alpha \in A;$

$U_t\ \in \tau\ \forall\alpha \cap X = U\ \alpha \in AU_i;$

$\Rightarrow \nexists, \alpha_1 \dots \alpha_\mathrm{n}, X = U_{\mathrm{i}=1}^{n} U\alpha i$

[1-16] Theorem: (1)

The continuous image of compact space is compact. i.e, $If(X,r) \rightarrow (Y,\tau')$ is continuous function and X is compact space. Then $f(x)$ is compact.

Proof: Let $f:(X,T) \rightarrow (y,T')$ be couhtinuous and X compact space. To aprove, $f(x)$ compact sit y
let $C = \{V_\alpha\}\alpha \in A$ open cover for $f(x)$

$$\Rightarrow f(x) \subseteq V_\alpha \in AV_\alpha, V_\alpha \in \tau' \forall \alpha \in A$$
$$\Rightarrow f^{-1}(f(x)) \subseteq f^{-1}(V_\alpha \in AV_\alpha)$$
$$\Rightarrow x \subseteq V_\alpha \in Af^{-1}(V_\alpha)(\text{ since } f^{-1}(f(x))) =$$

X And $f^{-1}(U_t \in AA_x) = U_t \in Af^{-1}(A_\alpha)$
since f is continuous $\Rightarrow f^{-1}(U_\alpha)\tau\ \forall x \in A$
$\Rightarrow \{f^{-1}(U_1)\}_{2\in A}$ is open cover for X
$\therefore X$ is compact$\Rightarrow \exists\ \alpha_1 \dots + \alpha_n;$

$$x \leq U_{i=1}^{n}\alpha \Rightarrow f(x) \subseteq f\left(U_{i=1}^{h} f^{-1}(v_{2i} i)\right)$$
$$\Rightarrow = U_{i=1}^{n}\left(f^{-1}(v_{\alpha i})\right)\ (\text{since } f(A \cup B) = f(A) \cup f(B))$$
$$\Rightarrow f(x) \leq U_{i=1}^{n} \forall\alpha I\ [\text{ since } f(f^{-1}(A)) \subseteq A)$$
$$\therefore f(x) \text{ compact Set}$$

[1-17] Theorem: (1) A closed subset of a compact space is compact.

Proof: let (X, τ) compact space and f closed set in X To prove. f compact set
let $C = \{U_\alpha\}\alpha \in A$ open cover of f

$$\Rightarrow f \subseteq U\alpha \in A\ U\sigma; U\alpha \in \tau\ \forall\alpha \in A$$
$$\therefore X = FUF^C \Rightarrow X - U_\alpha \in AU_\alpha F^c \text{ (since } F \subseteq U_{t\in A}U_\alpha)$$
$$\therefore U_\alpha \in \tau\forall\alpha \in A \cap F^c \in \tau \text{ (since } f \text{ closed set)}$$
$$\Rightarrow \{U\alpha\}\ \alpha{\in}A \cup \{F^c\} \text{ open cover of } X$$
$$\therefore x \text{ compact } \Rightarrow \exists\alpha_2 \dots \alpha_n, X = (U_{i=1}^n U\alpha i)\ F^c$$
$$\text{But}, F \subseteq X \Rightarrow F \subseteq (U_{i=1}^n U_{\alpha i}) \cup F^c$$
$$\text{since } F \cap F^c = \phi \Rightarrow F \subseteq U_{i=1}^n U_\alpha$$
$$\therefore \text{ F compact Set}$$

Notes that the condition being F closed is very important and the theorem is not true if the Condition deleted.

[1-18] Theorem: let (X, T) be a topological Space and P be the family of closed sets on x, then

(1) $x, \emptyset \in P$

(2) If$A \cdot B \in$ P. Then $A \cup B \in F\ \forall A \cdot B \in P$
(3) If$A_\alpha \in P;\ \tau \in A$, Theh $\cap_{r\in A} \in P\ \forall A_\varepsilon \in P$
proof: (1)

$$\therefore \phi \in \tau \Rightarrow \phi^c \in P \Rightarrow X \in P$$
$$\therefore X \in \tau \Rightarrow X^c \in P \Rightarrow \phi \in \emptyset$$

(2) let

$$APB \in F \Rightarrow A^c \cdot B^c \in \tau$$
$$\Rightarrow A^C \cap B^c\tau$$
$$\Rightarrow (A \cup B)^C \in \tau$$
$$\Rightarrow A \cup B \in P$$

(3) let

$A\alpha \in P \; \forall \alpha \in A$

$\Rightarrow \mathrm{A}_{\alpha}^{C} \in \tau \; \forall \alpha \in A$

$\Rightarrow U_{\alpha \in A} A_{\alpha}^{C} \in \tau$

$\Rightarrow (\cap_{\alpha \in A} A_{*})\mathrm{c} \; \in \tau$

$\Rightarrow \cap_{\alpha \in A} A_{\alpha} \in P$

Chapter two

Chapter two
Metric space

[2-1] Definition: "Metric space"(2)

If a set in which the concept of distance between the elements of the set it known, and it is called three-dimensional space or Euclidean space, as the Euclidean metric defines the distance between Points as a Straight Line Connecting them.

[2-2] Definition: "Metric spaces"(2)

Let X be a non-empty set and $d: XxX \to R$ is called the distance function satisfy the following conditions:

(a) $d(x,y) \geqslant 0$, for all $x, y \in X$

(b) $d(x,y) = 0$ if $x = y$

(c) $d(x,y) = d(y,x)$ for all $x, y \in x$

(d) $d(x+y) \leq d(x,z) + d(z,y)$ (Triangle inequality)

then (x,d) is called metric space.

[2-3] Example :(2)

Let $d: R \times R \longrightarrow R$ defined by $d(x,y) = |x-y|$, for all $x, y \in R$ Show that (R,d) is a metric space

solution:

(1) $d(x,y) = |x-y| > 0$, for all $x, y \in R (By$ def: of absolutely value)

(2) $d(x,y) = 0$

$$\Leftrightarrow |x-y| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$$

(3) $d(x,y)$

$$
\begin{aligned}
&= |x-y| \\
&= |-(y-x)| \\
&= |-1| - |y-x| = |y-x| = d(y,x)
\end{aligned}
$$

(4) $d(x,y)$

$$
\begin{aligned}
&= |x-y| \\
&= |x-z+z-y| \\
&\leq |x-z| + |z-y| \\
&\leq d(x,y) + d(z,y)
\end{aligned}
$$

$\therefore$ (R,d) is a metric space.

Open Balls and closed Bats and the Ball in metric Space. :(2)

[2-4] Definition: "open Bulls"

let(X,d) be a metric space, group $(Y \in K_{+})r)_{0,}X_0 \in X$

$$B\left(X_{0,\mathrm{r}}\right) = \{X \in X: d(X,x_0) < r\}$$

is called open Balls in (X,d) center (X_0) and
radius(r).

[2-5] Definition: "Closed Balls"

Let (X,d) be a metric space and Definition

$$B\left[x_{0,r}\right] = \{x \in X: d(x,x_0) \leq r\}$$

is called closed Balls in (x, d) center (x_0) and radius(r).

[2-6] Definition: "The Balls"

let $(X \cdot d)$ be ahretric Space and Definition

$$B\left(x_{0,r}\right) = \{x \in X: d(x,x_0) = r\}$$

is called Balls in (X,d) the center (X_0) and radius(r).

open set and closed Set in metric space

[2-7] Definition: "open set"

Let (X,d) be a metric space and $A \leq X$. Ais called ah open Set if $\forall P \in A$ there exists $r >_0$ $(r \in R)$ such that

Br (p) $\subseteq A$. i.e. A is open if $A^{\circ} = A$

[2-8] Example:

Let (R,d) be a metric Space, which of the following sets is open $A = (0,1)$ is open since $A^{\circ} = (0,1) = A$.

[2-9] Remark: every open interval in *R* is an open set

EX: $(a, \infty) \cdot (-\infty, a), (a, c)$ are opeh Sets.
Soll: $\forall b \neq a \cdot \exists d = |b - a|$

$$S.\text{t}(b - \epsilon, b + \epsilon)C(a, \infty)$$
$$\therefore (a, \infty)^{\circ} = (a, \infty)$$

$\therefore$ (a. ∞) is open set.

[2-10] theorem: Every ball is an open Set (2)

Proof: $Br(x_0) = \{x \in X: d(y, x_0) < r\}$

Let $y \in \text{Br}(x_0) \to d(y, x_0) = r_1 < r$ take $\epsilon = r - r_1 > 0$ T.P $B \in (y) \subseteq \text{Br}(x_0)$

$$\text{Let } z \in B \in (s) \text{ T.P. } z \in B_r(x_0)$$
$$d(z, y) < \epsilon \text{ T.P, } d(z, x_0) < r$$
$$d(z, x_0) \leq d(z, y) + d(y_1, x_0) < \epsilon + r_1$$
$$< r - r_1 + r_1 < r$$
$$\therefore d(z, x_0) < r \to Z \in Br(x_0)$$
$$\therefore B_{\in}(y) \leq Br(x_0)$$

Hence every point of $\text{Br}(X_0)$ is ah interial Point.
$\therefore \text{Br}(X_0)$ is an open set.

[2-11] Definition: Closed Set

let(X, d) be amtric space and $f \subset X$. is called an closet Set in (X, d) if The complement of this set X/f is an opeh set in (X, d).

i.e $f \leq X$ is closed if $F' \subseteq F$.

[2-12] Example :(2)

Let $(R \cdot d)$ be a metric space and $A = (0.1)$
soll:

$$\therefore A' = [0,1] \text{ and } A' \notin A$$
$$\therefore A = (,1) \text{ is not closed}$$

[2-13] Example:

Let (Rod) be a metric space and $A = [2,7]$
soll:

$$\therefore A' = [2.7] \text{ and } A' \leq A$$
$$\therefore A = [z.7] \text{ is closed}$$

[2-14] Theorem :(2)

in a metric space a set E is closed if and only if its
complement is open
proof: Suppose that E is closed Set
T.P. E^c is open

let $X \in E^c \Rightarrow X \notin E$

$\because$ E is closed
$\therefore$ X is hot atimit point of E

$\Rightarrow r > 0 \cdot$ s.t $\mathrm{Br}(x) \wedge E = \emptyset$
$\Rightarrow x \in \mathrm{Br}(x) \subseteq E^C \Rightarrow E^c$ is open

Suppose that E^c is open
T.P E is closed

Let X be a limit point of E

$\forall Br(x)$ sir. $Br(x) \wedge E \neq \emptyset$
$\therefore Br(x) \notin E^C$ $\therefore E^C$ is open
$\therefore x \notin E^C$
$\therefore x \in E$
$\therefore E$ is closed.

[2-15] Definition: "Compact Space" (2)

A Subset E of a metric space is called compact if every open cover of E contahins a finite sub cover.

i.e $E \leq U_{i\in I} Gi \Rightarrow \mathrm{I}]_{i_1}, i_2,$ in $s \cdot t \cdot E = U_{j=1}^{k} G_{ij}$

[2-16] Example :(2)
every finite set of a metric space is compact.

Solution: Let (X, d) be a metric space and $E \subseteq X$

where: $F = \{x_1, x_2, \dots, x_n\}$

Let $\{G_i\}_{i \in I}$ be an open cover of E
∴ E ∪ i ∈ IG_{i1}

Let $x_1 \in G_{i1}, x_2 \in G_{i2}, x_n \in G_{ih}$
∴ $E = \{x_1, x_2, \dots x_n\} \subseteq \cup_{j=1}^{n} G_{ij}$

∴ E is compact.

[2-17] Theorem :(2)
closed Subset of compact metric space (X, d) is Compact.
Proof: Let $E \subseteq X$ be closed.
let $\{G_i\}_{i \in I}$ be an open Covering for E

ie. E $\subseteq U_{i \in I} G_i$

∴ F is closed $\Rightarrow E^c$ is open
∴ $X = U_{i \in I} G_i \cup E^c$
∴ $\exists_{i_1}, i_2, i_n \in I$
S.t. $X = U_{i=1}^{n} G_i U E^c$
∴ $F \subseteq X \Rightarrow EGU_{i=1}^{n} G_i$

∴ E is compact.

Chapter Three

Chapter Three

Metric Topology

[3-1] Introduction :(3)

Metric topology is a fundamental concept in mathematical analysis and topology that extends the idea of distance in a space. It allows us to study the structure and properties of spaces based on a metric function. In this chapter, we explore deeper aspects of metric topology, including completeness, compactness, connectedness, and real-world applications.

Metric topology is a type of topology that arises from a **metric space**. A **metric space** is a set X equipped with a **metric (distance function)** $d: X \times X \to R$, which satisfies the following properties for all $x, y, z \in X$:

1. **Non-negativity**: $d(x, y) \geq 0$ (distance is always non-negative).
2. **Identity of Indiscernible**: $d(x, y) = 0$ if and only if $x = y$.
3. **Symmetry**: $\mathrm{d}(x, y) = d(y, x)$.
4. **Triangle Inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

Using this metric, we define a **topology** on X by specifying the open sets:

- A subset $U \subseteq X$ is **open** if, for every point $x \in U$, there exists an **ϵ-ball** around x that is completely contained in U.
- The **ϵ-ball** around x is defined as:

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

Where $\boldsymbol{\epsilon > 0}$ is a positive real number.

This collection of open sets forms a **topological space**, called the **metric topology**.

[3-2] Examples of Metric Topology :(3)

1. Euclidean Space R^n

- The standard distance is $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
- The open sets in the metric topology are just the usual open balls in R^n.

2. Discrete Metric

- Define $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$.
- Every subset of $\boldsymbol{X}$ is **open**, so the metric topology is the **discrete topology**.

3. Taxicab Metric (Manhattan Distance)

- Defined on R^2 by $d\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$.
- The topology differs from the usual Euclidean topology since open balls look like **diamonds** instead of circles.

[3-3] Properties of Metric Topology: (3)

1. Completeness

A metric space (X, d) is complete if every Cauchy sequence converges to a point in X. A sequence $\{x_n\}$ is Cauchy if:

$$\forall \epsilon > 0, \exists n \in k \text{ such that } m, n \geq k \Rightarrow d(x_m, x_n) < \epsilon.$$

- Example: The real numbers R with the Euclidean metric are complete.
- The space Q (rationals) is not complete since sequences like $x_n = (1 + 1/n)$ converge to an irrational number not in Q.
- Application: Completeness is essential in functional analysis and Banach spaces.

2. Compactness

A subset S of a metric space (X, d) is compact if every open cover has a finite subcover. Equivalent conditions for compactness:

- Every sequence in S has a convergent subsequence (Sequential Compactness).
- S is bounded and closed (Heine-Borel Theorem, only in R^n.
- Application: Compactness is crucial in optimization and theoretical physics (e.g., compact manifolds in relativity).

3. Connectedness

A metric space is **connected** if it is not the union of two disjoint nonempty open sets.

- A space is **path-connected** if any two points can be connected by a continuous path.
- **Example**: The interval [0,1] is connected, but $(0,1) \cup (2,3)$ is disconnected. Application: Used in graph theory, circuit design, and network topology.

[3-4] Conclusion:

Metric topology provides a bridge between **pure mathematics and applied sciences**. By studying properties like completeness, compactness, and connectedness, we gain insights into **geometry**, **physics**, **computer science**, **and engineering**. This chapter illustrates how mathematical abstraction plays a critical role in solving real-world problems.

References

1. Thtroduction to general topology"（Revised）K． D． Joshi ihstitute of technology Bombay.
2. An introduction to general topology ˝ PaylE. Lang, whitersity of Arkahsas.
3. Metric Space－micheúl Ó Searcoid. UCD School of mathematical Sciehces, University College pubih Belfield Eveland.

١. الطوبولوجيا العام تأليف احمد عبد القادر رمضان طه مرسي العدوي - قسم الرياضيات - كلية العلوم - جامعة الملك سعود - فرع القصيم.
٢. اساسيات الطوبولوجيا العامة - تأليف وليام بيدفن - ترجمة عطا الله ثامر العاني.
٣. الطوبولوجيا للعامة - المنجى بلال (١٩) جانفي ٢٠٢٠
٤. مقدمة في الطوبولوجيا العامة تأليف الدكتور سمير بشير حديد - كلية التربية - جامعة الموصل.