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On WEAKLY REGULAR RINGS

A Thesis Submitted to the council of college of Education/ University of Misan in Partial Fulfillment of the Requirements for the Degree of Bachelor in mathematics.

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بِسْمِ اللهِ الرَّحمَنِ الرَّحِيمِ

((وَلِيَعْلَمَ الَّذِينَ أُوتُوا الْعِلْمَ أَنَّهُ الْحَقُّ مِن رَّبِّكَ فَيُؤْمِنُوا بِهِ فَتُخْبِتَ لَهُ قُلُوبُهُمْ وَإِنَّ اللَّهَ لَهَادِ الَّذِينَ آمَنُوا إِلَى صِرَاطٍ مُّسْتَقِيمٍ))

صدق الله العلي العظيم

سورة الحج أية 54

الإهداء

إلى من ظهوره سيملأ الأرض قسطاً وعدلا إلى الإمام الحجة ابن الحسن (عجل الّله فرجه)..

الى كل من كان لهم الفضل عليّ من بعد ربي الذين ساهموا باكمال هذه المسيرة التي طالت وتكاد تختم بالنجاح..

إلى أبي العزيز والظل الظليل الذي يرقب وصولي للمنى والى الرؤوم التي لم تبرح تلازمني في اليسر والعسر وفي الصغيرة والكبيرة أمي الحبيبة والى اخوتي الذخر والناصحون

الى اساتذتي الكرام الذين لم يبخلوا عليّ من علمهم وسقوني من عصارة جهدهم علماً نافعا ومبادِئ تبقى عالقة في الاذهان

الى اصدقائي المخلصين والاحبة وكل من ساهم ولو بكلمة طيبة

اهديهم جميعاً هذا البحث تعبيراً عن امتناني لهم

الباحثان

الشكر والتقدير

في البداية الشكر للّه وله الحمد في الاولى والاخرة فله الفضل كله في اتمام هذا العمل.

واتقدم بالشكر بعد رب العالمين الى صاحبة الفضل الاستاذة المتفانية والحريصة الى الدكتورة (هبة ربيع بعنون)

واني لاعجز عن شكرها وامتناني لحضرتها فلم تقصر ابدا بل كانت داعمة ومحفزة ومعطاءة فكل الأحرف تعجز عن شكرها فلولاها لم يكتمل هذا العمل

والشكر موصول لاساتذتي في قسم الرياضيات فقد كان فضلهم علي كبيراً لولاهم ما وقفت الان بين ايديهم ملقيا عليهم هذا العمل وكلي رجاء ان يتقبلوا مني هذا اليسير

Supervisor approval

I certify that this research **(On weakly regular rings)** submitted by the students**, (Saber shaker saber and Sahar Mohamed Yasser)**, took place under my supervision at the University of Misan/ College of Education/ Department of Mathematics. It is part of the requirements for obtaining a bachelor's degree in the College of Education / Department of Mathematics.

Supervisor

Signature:

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Date: / / 2025

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ABSRACT

Let R be a commutative ring with identity. This work studies a class of rings called weakly regular as a generalization of regular ring, , with some examples and properties.

INRODUCTION

Throughout, all rings are commutative with identity. In this work, we will study the concepts of the weakly regular rings, which is a generalization of regular rings. This research has two chapters:

In chapter one, we recall some properties about the rings.

In chapter two, study the weakly regular rings with examples and properties.

CHAPTER ONE

(BACK GROUND on RINGS)

Chapter one: Back ground on rings

In this chapter we will recall the definition of rings and ideals with some examples and properties. See [1].

Definition (1.1): A ring is an ordered triple $(R\ ,+,\cdot)$ consisting a nonempty set R with two binary operations $+$ and $\cdot$ defined on R such that

1) $(R,+)$ is a commutative group,
2) $(R,\cdot)$ is a semigroup, and
3) The operation $\cdot$ is distributive over the operation $+$.

Definition (1.2):

1) A commutative ring is a ring $(R\ ,+,\cdot)$ in which multiplication is a commutative operation, $a \cdot b = b \cdot a$ for all $a, b \in R$.
2) A ring with identity is a ring $(R\ ,+,\cdot)$ in which there exists an identity element for the operation of multiplication, normally represented by the symbol 1, so that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$.

Example (1.3): The mathematical system $(\mathbb{Z}\ ,+,\cdot)$ is a a commutative ring with identity 1 , since

1) $(\mathbb{Z}\ ,+)$ is a commutative group ,
2) $(\mathbb{Z}\ ,\cdot)$ is a semigroup, and
3) $\forall a, b, c \in \mathbb{Z} \quad a\,.(b + c) = (a\,.b) + (a\,.c)$

   And $(b + c)\,.a = (b\,.a) + (c\,.a)$

4) The multiplication is a commutative operation with identity 1.

<u>Definition (1.4):</u>

Let $(R ,+,\cdot)$ be a ring and let $\emptyset \neq\ I \subseteq R$, then $(I ,+,\cdot)$ is called an ideal of $(R ,+,\cdot)$ if and only if

1) $a - b \in\ I, \forall a, b \in I$,
2) $\forall r \in R$ and $\forall\ a \in I$ then $ar\ \in I$ and $ra\ \in I$.

- If $ar \in I$ for all $r\ \in R$, $a \in I$ then I is called right ideal.
- If $ra\ \in I$ for all $r \in R$, $a\ \in I$ then I is called left ideal.
- If I is both right and left ideal then I is called two sided ideal or simply ideal.

Example(1.5): The set of all even integer number $(\mathbb{Z}_e\ ,+,\cdot)$ is an ideal of the ring $(\mathbb{Z} ,+,\cdot)$.

<u>Definition (1.6)</u>: Let $(I ,+,\cdot)$ and $(I ,+,\cdot)$ be ideals of the ring $(R ,+,\cdot)$. Then $I \cdot J = \{\sum_{i=1}^{n} a_i b_i | a_i \in I\ and\ b_i \in J\}$ is also ideal of the ring $(R ,+,\cdot)$.

<u>Definition (1.7)</u>:

An ideal $(I ,+,\cdot)$ of the ring $(R ,+,\cdot)$ that generated by a single element $a \in R$ is called principal ideal, we denoted by $\langle a \rangle$ or Ra and aR which defined by $I =\ \langle a \rangle\ = \{r.a : r\ \in\ R\} = Ra$.

Example (1.8): In the ring $(\mathbb{Z}_6\ ,+_6\ ,\cdot_6\ )$, the all ideals of this ring are principal: $\langle \bar{0} \rangle = \{\bar{0}\}$, $\langle \bar{1} \rangle = \mathbb{Z}_6$, $\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}\}$ and $\langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$.

Definition (1.9): A ring $(R, +, \cdot)$ is said to have zero divisors, if there exists nonzero elements $a, b \in R$, such that $ab = 0$.

Examples (1.10) :

1) The rings $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ which does not have zero divisors.
2) The ring $(\mathbb{Z}_6, +_6, \cdot_6)$, has zero divisor since $\bar{2} \neq \bar{0}$ and $\bar{3} \neq \bar{0}$ but $\bar{2} \cdot \bar{3} = \bar{0}$.

Definition (1.11): An integral domain is a commutative ring with identity which does not have divisors of zero.

Example (1.12): The rings $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are integral domains.

Definition(1.13) : Let $(R, +, \cdot)$ be a ring and $(I, +, \cdot)$ be an ideal of R. Then the set $\frac{R}{I} = \{r + I | r \in R\}$ with the following operation

$$(a + I) + (b + I) = ab + I$$

and

$$(a + I) \cdot (b + I) = ab + I$$

Be a ring which called quotient ring.

Definition(1.14) :

A non zero ideal $(I,+,\cdot)$ of a ring $(R,+,\cdot)$ is called Maximal ideal. If $I \neq R$ and there exists no proper ideal of a ring R containing I, i.e. I is a maximal ideal of R if $I \neq R$, and if there exists an ideal $(J,+,.)$ in R with $I \subset J \subseteq R$, then $J = R$.

Example (1.15): In the ring $(\mathbb{Z}_{12}, +_{12}, ._{12})$ the following ideals are proper

1) $I_1 = \langle\bar{2}\rangle = (\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\},+_{12},._{12})$.
2) $I_2 = \langle\bar{3}\rangle = (\{\bar{0},\bar{3},\bar{6},\bar{9}\},+12,._{12})$.
3) $I_3 = \langle\bar{4}\rangle = (\{\bar{0},\bar{4},\bar{8}\},+_{12},._{12})$.
4) $I_4 = \langle\bar{6}\rangle = (\{\bar{0},\bar{6}\},+_{12},._{12})$

I_1 and I_2 are maximal ideals in a ring $\mathbb{Z}_{12}$, since there is no a proper ideals of a ring $\mathbb{Z}_{12}$ containing I_1 and I_2.

But I_3 is not maximal ideal in $\mathbb{Z}_{12}$, since $\langle\bar{4}\rangle \subset \langle\bar{2}\rangle$ and $I_4 =$ is not maximal ideal in $\mathbb{Z}_{12}$, since $\langle\bar{6}\rangle \subset \langle\bar{2}\rangle$.

Definition(1.16):

A Boolean ring $(R,+,.)$ is a ring with identity and every element in

a ring R is an idempotent element, That is $a^2 = a$, $\forall a \in R$.

Example (1.17): A ring $(\mathbb{Z}_2,+_2,\cdot_2)$ is a Boolean, since $\mathbb{Z}_2 = \{\bar{0},\bar{1}\}$

$\bar{0}^2 = \bar{0}$ and $\bar{1}^2 = \bar{1}$.

Definition (1.18): A ring $(F,+,.)$ is said to be field if and only if $(F,+,.)$ is a commutative ring with identity and every a non zero element $a \in F$ is invertible element**.**

Example (1.19): The rings $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are filed.

Definition (1.20): A ring $(\mathrm{R},+,.)$ is called directly decomposable if $\mathrm{R}=0$ or there is a proper ideal I of R such that $I+J=R$ and $I\cap J=\{0\}$ for some ideal J of R.

CAPTER TWO

Weakly Regular Rings

Chapter Two: Weakly Regular Ring

In this chapter, we will recall the concept of the weakly regular rings with some properties. See [2] and [3].

Definition(2.1):

- A ring R is called left weakly regular, if $B^2 = B$ for each left ideal B of R.
- A ring R is called right weakly regular, if $B^2 = B$ for each right ideal B of R.
- A ring R is called weakly regular, if $B^2 = B$ for each ideal B of R.

Examples and Remarks (2.2):

1) Every regular ring is weakly regular ring.

   Proof: Let R be a regular ring and I be an ideal of R. Then $II \subseteq I$, let $a \in I \subseteq R$, so there exists $b \in R$ such that $a = aba$. Since I is an ideal, hence $ab \in I$ and $aba \in II$. So $a \in II$ and $I \subseteq II$. Thus, $II = I$ and R is weakly regular ring.

2) The weakly regular ring need not to be regular in general. For example, let $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\}$ is weakly regular ring but not regular. For each $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, there exists $X = \begin{bmatrix} a^{-1} & \frac{b-2ab\,c^{-1}}{a^2} \\ 0 & c^{-1} \end{bmatrix} \in R$ such that $A^2X = A \in ARAR$.

3) The ring $(\mathbb{Z}_2, +_2, ._2)$ is weakly regular, since the all ideals of $\mathbb{Z}_2$ are: $\langle\bar{0}\rangle$ and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ with $\langle\bar{0}\rangle\langle\bar{0}\rangle = \langle\bar{0}\rangle = \{\bar{0}\}$, $\mathbb{Z}_2\mathbb{Z}_2 = \mathbb{Z}_2$. In general, $(\mathbb{Z}_n, +_n, ._n)$ is always weakly regular ring when n is a prime number.
4) The ring $(\mathbb{Z}, +, .)$ is not weakly regular, since the ideal $\langle 2\rangle\langle 2\rangle = \langle 4\rangle \neq \langle 2\rangle$.

Proposition(2.3): The following conditions are equivalent for any ring R

a) R is right weakly regular.
b) For every $r \in R$, $r \in (rR)^2$
c) For any two right ideals $K_1 \subseteq K_2$, of R , $K_1K_2 = K_1$.

Proof:

$(\boldsymbol{a}) \Longrightarrow (\boldsymbol{b})$ Let $\in R$ and K be the right ideal generated by r. Then $r \in K = K^2, K = rR$. Therefore, $r \in (rR)^2$.

$(\boldsymbol{b}) \Longrightarrow (\boldsymbol{c})$ Taking $r \in K_1$ we have $r \in (rR)^2 \subseteq K_1K_2$. Hence $K_1 \subseteq K_1K_2$. Since K_1 is right ideal, so $K_1K_2 \subseteq K_1$. Therefore, $K_1K_2 = K_1$.

$(\boldsymbol{b}) \Longrightarrow (\boldsymbol{c})$ Let I be a right ideal. So $I \subseteq I$ and $II = I$. Thus, R is right weakly regular.

Proposition(2.4): If A is proper two sided ideal of a right weakly regular ring R, then each element of A is left zero divisor .

Proof: If the element x of A is not a left zero divisor, then x is invertible and

equation $xR = xRxR$ yields. $R = RxR \subseteq A$ which is a contradiction.

Proposition(2.5): The centre of any right weakly regular ring is regular.

Proof: If $x \in C$, the centre of the right weakly regular ring R, then $x \in (xR)^2 \subseteq x^2R$ implies immediately that $x = x(x^k y)x$ with y $\in R$ and k $\geq$ 1. Moreover for every $z \in R$, $z(x^k y) = x^{k-1}(xz)y = x^{k-1}(x^{k+1}yx)zy = x^{k-1}yz(x^{k+1}yx) = x^{k-1}y(zx) = (x^k y)z$, that mean $x^k y \in C$.

Remark (2.6) [3]: If R is a ring with identity, then the following conditions are equivalent:

a) R is right weakly regular.
b) If I is a right ideal and K is a two sided ideal of R then $I \cap K =$

IK.

c) If I and J are two right ideals of R, then $\cap J = IJ$

References

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[3] V.S. Ramamurthi, "Weakly regular rings", Canadian Mathematical Bulletin, Vol.16, no.3, (1973): 317-321.