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COFINITELY MODULES

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بسم الله الرحمن الرحيم

﴿هُوَ الَّذِي جَعَلَ الشَّمْسَ ضِيَاءً وَالْقَمَرَ نُورًا وَقَدَّرَهُ مَنَازِلَ لِتَعْلَمُوا عَدَدَ السِّنِينَ
وَالْحِسَابَ مَا خَلَقَ اللَّهُ ذَلِكَ إِلَّا بِالْحَقِّ يُفَصِّلُ الْآيَاتِ لِقَوْمٍ يَعْلَمُونَ﴾

صدق الله العظيم

(سورة يونس-آية ٥)

الاهداء

إلى والدتي الغالية التي لم تأل جهدا في تربيتي وتوجيهي أقدم هذا العمل

إلى سبب وجودي في الحياة. . والدي الحبيب لك كل التجلي والاحترام

الشكر والامتنان

بعد رحلة بحث وجهد واجتهاد تكللنا بإنجاز هذا البحث نحمد الله عز وجل

على النعمة التي من بها علينا فهو العلي القدير

كما لا يسعنا الا ان نحظى بأسمى عبارات الشكر والتقدير إلى (الاستاذ عبد الكريم)

لما قدمه لنا من جهد ونصح ومعرفه طيلة انجاز هذا البحث.

فله منا كل الشكر

Supervisor approval

I certify that this research **(cofinitely modules)** submitted by the student, **(Noor AL-Hoda Muslim Taher)**, took place under my supervision at the University of Misan / College of Education / Department of Mathematics. It is part of the requirements for obtaining a bachelor's degree in the College of Education / Department of Mathematics.

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ABSTRACT

Let R be any ring with identity and Let M be aunitary Left R-module. This research Studies Two Types of modules. The first is supplemented modules and the second cofinitely supplemented modules with some examples and properties.

INTRODUCTION

Throughout all rings are associative with identity and modules are rings unitary. In This work. we will study the concepts of supplemented modules and the second cofinitely supplemented modules this research has Two chapters:

In chapter One, we recall the definition of the group modules and Some Properties about small submodules

In chapter Two, there are Two sections. In section one Study supplemented modules with example and Properties

Section Two Studies The cofinitely supplemented modules with examples and properties.

CHAPTER ONE

BASIC CONCEPTS OF MODULES

Chapter one

Basic concepts of modules

In This chapter we will recall the definition of the group, modules and small submodules with some examples and properties.

Definition (1.1):[1]

A group is an ordered Pair (G,$*$) consisting anon empty set G and binary operation $*$ defined on G satisfy the following.

1. G is a closed under $* \rightarrow a * b \in G, \forall a * b \in G$
2. $*$ is assoca i tive on G$\rightarrow$ a (b$*$c) = (a $*$b) $*$c $\forall a$, b, c $\in$ G
3. $\exists e \in G$. such that $\forall a \in G$. $a * e = e * a = a$

where a^{-1} is called the inverse element of a

Example (1.2): (Z,+) is a group

Solution:

1.Z is a closed under (+) a+b $\in$ z.$\forall a$, b $\in$ z

2. + is associative on z$\rightarrow$$a$ + (b + c) = (a+b) + c $\forall a$,b , C $\in$ Z

3. 0 is the identity element with add. since $\forall a \in Z \rightarrow a + 0 = 0 + a = a$

4.$\forall a \in Z.\exists - a \in Z$. such that a + (-$a$) = (-$a$) + a = 0

Definition (1.3): [1]

A group (G. $*$) be a called commutative group or (abelian group) if

$a * b = b * a \ \forall a$, b $\in$ Z

Example (1.4): show that (Q-{0} ,.) is a commutative group.

Solution:

1. Q-{0} is a closed under (·) , since ∀a,b∈Q-{0}→ a+b ∈ Q-{0}
2. (·) is associative on Q-{0}. Since (a ·b)·c = a ·(b·c)∀a, b, c ∈ Q-{0}
3. I is the identity element with multiply of Q-{0} since ∀a ∈ Q-{0}→ a ·1=1· a = a
4. ∀a ∈ Q-{0}· ∃ a^{-1}∈ Q-{0}. Such that $a.a^{-1}$=a^{-1}· a = l where a^{1} = 1/a ∈ Q-{0}
5. ∀a ,b ∈ Q-{0}→ a ·b = b· a →(·) is a commutative on Q-{0}

<u>Definition (1.5):</u> [1]

Let R be a non-empty set and let ∗, # be two binary operations on R. Then (R, ∗, #) is a ring if:

i. (R, ∗) is an abelian group

1. Closed because ∀a, b ∈ R then a ∗ b ∈ R
2. Associative because ∀a, b, c ∈ R then a ∗ (b ∗ c) = (a ∗ b) ∗c
3. ∀a ∈ R, ∃e ∈ R s.t a ∗ e = e ∗ a = a
4. ∀a ∈ R,∃ a^{-1} ∈ R s.t a ∗ a^{-1} = a^{-1} ∗ a = e
5. commutative because ∀a, b ∈ R then a ∗ b = b ∗ a

ii. (R, #) semi group.

1. Closed because ∀a, b ∈ R then a#b ∈ R
2. Associative because ∀a, b, c∈ R then a#(b#c) = (a#b) #c

iii. Distributive

1. ∀a, b, c ∈ R then a#(b ∗ c) = a#b ∗ a#c

Example (1.6):

($\mathbb{Z}$, +,·) is a ring

i. ($\mathbb{Z}$, +) is an abelian group

1. Closed because $\forall a$, b $\in \mathbb{Z}$ then a + b $\in \mathbb{Z}$
2. Associative because $\forall a$, b, c $\in \mathbb{Z}$ then a + (b + c) = (a + b) + c
3. $\forall a \in \mathbb{Z}$. $\exists e \in \mathbb{Z}$ s.t $a + e = e + a = a$, $e = 0$
4. $\forall a \in \mathbb{Z}$. $\exists\, a^{-1} \in \mathbb{Z}$ s.t $a + a^{-1} = a^{-1} + a = e$, $a^{-1} = a$
5. Commutative because $\forall a$, b $\in \mathbb{Z}$. then a + b = b + a

ii (Z, ·) semi group

1. Closed because $\forall a$, b $\in \mathbb{Z}$ then $a \cdot$ b $\in \mathbb{Z}$
2. Associative because $\forall a$, b, c $\in$ Z than $a \cdot$ (b · c) = ($a \cdot$ b) · c

iii. Distributive

1. $\forall a$, b, c $\in \mathbb{Z}$ then $a \cdot$ (b + c) = $a \cdot$ b + $a \cdot$ c

$\therefore$ ($\mathbb{Z}$, +, ·) is a ring

<u>**Definition (1.7):**</u> [1]

Let R be a ring with identity an abelian group (M, +) is called a left R- module (or left R-module over R) if there exists a mapping f: R×M→M such that f(r,m) = r·m$\forall$r $\in R$ and $\forall$m $\in M$ satisfying the following conditions

1. f(r, $m_1 + m_2$) = f(r, m_1) + f(r, m_2) or r ($m_1 + m_2$) = r·m_1+ r·m_2 $\forall$r $\in R$ $\forall m_1, m_2 \in M$
2. f($r_1 + r_2$, M) = f(r_1, m) +f(r_2, m) or (r_1, r_2) m = r_1m + r_2m $\forall$ r_1, $r_2 \in R$,$\forall$m $\in M$
3. f(r_1r_2, m) = f(r_1),f(r_2, m) or (r_1r_2) m= r_1(r_2m)

   $\forall$ r_1, $r_2 \in R$ and $\forall$m $\in M$
4. If in addition 1.m = m $\forall$m $\in M$, then M is called a unital R-module

Example (1.8): Q is Z-module

Solution:

Since $\mathbb{Q}$ is an abelian group and the ring Z has unity then $\exists$ amapping

$f: \mathbb{Z} \times \mathbb{Q} \rightarrow \mathbb{Q}$ s.t

$a(r, m) = r \cdot m \ \forall r \in \mathbb{Z}$ and $\forall m \in \mathbb{Q}$. Let $r_1, r_2 \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{Q}$ then

1. $f(r, m_1 + m_2) = r\cdot(m_1 + m_2) = (m_1 + m_2) + (m_1 + m_2) + \ldots\ldots + (m_1 + m_2)$

$f(r, m_1 + m_2) = (m_1 + m_1 + \ldots\ldots + m_1) + (m_2 + m_2 + \ldots\ldots + m_2) = r\cdot m_1 + r\cdot m_2$

2. $f(r_1 + r_2, m) = (r_1 + r_2)\cdot m = m + m + \ldots\ldots + m + m + m$

$f(r_1 + r_2, m) = (m + m + \ldots\ldots + m) + (m + m + \ldots\ldots + m) = r_1\cdot m + r_2\cdot m$

3. $a(r_1\cdot r_2, m) = (r_1\cdot r_2)\cdot m = m + m + m + \ldots\ldots + m + m + m \rightarrow *$

$r_1\cdot(r_2\cdot m) = r_1\cdot(m + \ldots\ldots + m + m)$

$r_1\cdot(r_2\cdot m) = (m + \ldots\ldots + m + m) + (m + \ldots\ldots + m + m + \ldots\ldots + (m + \ldots\ldots + m + m)$

$r\cdot(r_2\cdot m) = m + m + m + \ldots\ldots + m + m + m \rightarrow **$from$*and**$we get $(r_1\cdot r_2)\cdot m = r_1\cdot(r_2\cdot m)$

4. Since 1 is the unity of a ring $\mathbb{Z}$, then $1\cdot m = m$.

Therefore, by module definition we get $\mathbb{Q}$ is $\mathbb{Z}$-module

<u>Definition (1.9):</u> [1]

Anon-empty subset N of R-module M is called a submodule of M iff

1.(N, +) is a subgroup of (M,+)
2. $r\cdot N \subseteq N \cdot \ \forall \ r \in R$

Example (1.10): $(\mathbb{Z},+)$ is submodule of module (Q,+) over a ring $(\mathbb{Z},+)$, since $\emptyset \neq \mathbb{Z} \subseteq Q$

1. $a+b \in \mathbb{Z}, \forall \ a, b \in \mathbb{Z}$

2.r· $a \subseteq \mathbb{Z}$· ∀ r ∈ R, ∀$a \in \mathbb{Z}$

<u>Definition (1.11):</u> [1]

A submodule N of a left - R- module M is said to be a direct summand of M if there is a sub module of M such that M=N⊕K. In other word there is a submodule k of M such that M=N+K and N∩K=0

Example (1.12): Let M=Z_6 as a left Z-module the direct summands of M is

Solution:

M, 0, N_1=<2> = $\{\bar{0}, \bar{2}, \bar{4}\}$ and N_2=<3> = $\{\bar{0}, \bar{3}\}$ are all direct summands of M

<u>Definition (1.13):</u> [1]

Let R and S be rings. Then a ring homomorphism ρ:R→S is a mapping for which for all r_1, r_2∈R we have

1. $\rho(r_1 + r_2) = \rho(r_1) + \rho(r_2)$
2. $\rho(r_1.r_2) = \rho(r_1)\cdot \rho(r_2)$

<u>Definition (1.14):</u> [1]

Let R and S be rings. A function ∝: R→S is called ring epimorphism if satisfies the following

1. ∝ is a homomorphism.
2. ∝ is surjective (onton).

<u>Definition (1.15):</u> [1]

Let M be an R-module M is called semi simple module if every submodule of M is a direct summand M.

<u>Definition (1.16):</u> [1]

Let G be a group. and A, B, C be subgroups of G the modular law states that.

$(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$

<u>Definition (1.17):</u> [1]

A module M is called duo. if every submodule of M is fully invariant.

<u>Definition (1.18):</u> [2]

Let M be an R-module and S be a sub module of M. S is said to be small submodule of M (denoted $s \ll M$) if for any submodule N of M such that $M = S + N$ We have $N = M$

Examples (1.19):

(1) For any R-module M, $\{\bar{0}\}$ is a small submodule of M but M is not small in M. Since the only case that we have is $0 + M = M$

(2) The submodule $\{\bar{0}, \bar{2}\}$ of the $\mathbb{Z}_4$ as $\mathbb{Z}$- module is small of $\mathbb{Z}_4$. Since the only case that we have is $\{\bar{0}, \bar{2}\} + \mathbb{Z}_4 = \mathbb{Z}_4$

(3) The submodules $\{\bar{0}, \bar{2}, \bar{4}\}$ of the $\mathbb{Z}_6$ as $\mathbb{Z}$-module is not small of $\mathbb{Z}_6$ since $\{\bar{0}, \bar{2}, \bar{4}\} + \{\bar{0}, \bar{3}\} = \mathbb{Z}_6$ but $\{\bar{0}, \bar{3}\} \neq \mathbb{Z}_6$

(4) In $\mathbb{Z}$ as a $\mathbb{Z}$-module, $2\mathbb{Z}$ is not a small submodule since $2\mathbb{Z}+3\mathbb{Z}=\mathbb{Z}$ but $3\mathbb{Z}\neq\mathbb{Z}$

Proposition (1.20):

(1) Let K_1 and K_2 be a submodule of R-module M $K_1 \ll$ M and $K_2 \ll$ M, then $K_1 + K_2 \ll$ M

(2) Let $f: M \to M'$ be an R-epimorphism and A $\ll$M. then $f(A) \ll M'$.

(3) Let M be an R-module M and K_1 and K_2 be a submodules of M with $K_1 \leq K_2 \leq$ M if $K_2 \ll$ M, then $K_1 \ll$M.

(4) Let M be an R-module and N, K, L are submodules of M With N $\subseteq$ K $\subseteq$ L $\subseteq$ M, if K $\ll$ L then N $\ll$ M.

Definition (1.21):

Let M be an R-module and N is submodule of M is said to be coffinite if $\frac{M}{N}$ is finitely generated.

CHAPTER TWO

COFINITELY SUPPLEMENTED MODULES

Chapter Two
supplemented and cofinitely supplemented modules

In This chapter, we will recall the concepts of the Supplemented and cufinitely supplemented modules with some examples and properties.

1-SUPPLEMENTED MODULE

In This section we recall the supplemented modules with some properties see [4] and [5].

Definition (2.1.1):

Let M be any R-module and N, K are submodules of M. N is called supplement of K in M if M = N + K and $N \cap K \ll N$. If every submodule of M has supplement then M is called supplemented module.

Example (2.1.2):

(1) Consider the module $\mathbb{Z}_4$ as $\mathbb{Z}$-module. Then $\mathbb{Z}_4$ is a supplement of a submodule $\{\bar{0}, \bar{2}\}$ in $\mathbb{Z}_4$ as $\mathbb{Z}$-module since $\{\bar{0}, \bar{2}\} + \mathbb{Z}_4 = \mathbb{Z}_4$ and $\{\bar{0}, \bar{2}\} \cap \mathbb{Z}_4 = \{\bar{0}, \bar{2}\} \ll \mathbb{Z}$ But the con verse is not true $\{\bar{0}, \bar{2}\}$ is not a supplement of $\mathbb{Z}_4$ since $\{\bar{0}, \bar{2}\} + \mathbb{Z}_4 = \mathbb{Z}_4$ and $\{\bar{0}, \bar{2}\} \cap \mathbb{Z}_4 = \{\bar{0}, \bar{2}\}$ is not small in $\{\bar{0}, \bar{2}\}$

(2) consider the module $\mathbb{Z}_6$ as-$\mathbb{Z}$-module $\{\bar{0}, \bar{2}\}$ is a supplement of $\{\bar{0}, \bar{2}, \bar{4}\}$ in $\mathbb{Z}_6$

Proposition (2.1.3):

Let M be a supplemented module and $N \subseteq M$ then $\frac{M}{N}$ is a supplemented.

Proof:

Let $\frac{K}{N} \subseteq \frac{M}{N}$,to prove $\frac{K}{N}$ has supplement in $\frac{M}{N}$. K ⊆ M, since M is supplemented, then there exists L ⊆ M such that M = K + L, and K ∩ L ≪ L, now $\frac{M}{N} = \frac{K+L}{N} = \frac{K}{N} + \frac{L+N}{N}$, to prove $\frac{K}{N} \cap \frac{L+N}{N} \ll \frac{L+N}{N}$, let $(\frac{K}{N} \cap \frac{L+N}{N}) + \frac{V}{N} = \frac{L+N}{N}$, to prove $\frac{V}{N} = \frac{L+N}{N}$, $\frac{K\cap(L+N)}{N} = \frac{N+(K\cap L)}{N}$, (by modular law). Then $\frac{N+(K\cap L)}{N} + \frac{V}{N} = \frac{L+N}{N}$ and N + (K ∩ L) + V = L+N, since N ⊆ V. Then (K ∩ L) + V = L + N hence K ∩ L ≪ L ⊆ L + N and K ∩ L ≪ L + N, therefore V= L + N and $\frac{V}{N} = \frac{L+N}{N}$.

Proposition (2.1.4):

Let M = $M_1 \oplus M_2$ be ado module, N and L are submodules of M_1, if N is a supplement of L in M_1 then N⊕M is supplement of L in M.

Proof:

Let N be supplement of L in M_1, then M_1 = N + L and N ∩ L ≪ N, since M = $M_1 \oplus M_2$, then M = (N + L) ⊕ M hence M = L + (N ⊕ M_2) but (N ⊕ M_2) ∩ L= (N ⊕ M_2) ∩ M_1 ∩ L_1= N ∩ L ≪ N.

then N ∩ L ≪ N ⊕ M_2, hence N ⊕ M_2 is a supplement of L in M

Proposition (2.1.5):

Let M be any R-module and V , U are submodules of M, V is supplement of U in M, then $\frac{V+L}{L}$ is supplenient of $\frac{U}{L}$ in $\frac{M}{L}$ for L ⊆ U.

proof:

Since V is a supplement of U in M. Then M = U + V and U ∩ V ≪ V for L ⊆ U we have U ∩ U(V + L) = (U ∩V) + L (by modular law) and $\frac{U}{L} \cap (\frac{V+L}{L}) = \frac{(U\cap V)+L}{L}$, since U ∩ V ≪ V, it follows that $\frac{(U\cap V)+L}{L} \ll \frac{V+L}{L}$. Now $\frac{M}{L} = \frac{U+V}{L} = \frac{U}{L} + \frac{V+L}{L}$, therefore $\frac{V+L}{L}$ is supplement of $\frac{U}{L}$ in $\frac{M}{L}$.

Proposition (2.1.6):

Let M be an R-module. If A is a supplement submodule in M. Then $\frac{A}{N}$ is a supplement submodule in $\frac{M}{N}$, where N is submodule of A.

Proof:

Since A is supplement in M. Then there exists submodule K of M. Such that $A + k = M$, and $A \cap K \ll A$. Now we have $\frac{A}{N} + \frac{K+N}{N} = \frac{M}{N}$ to show $\frac{A}{N} \cap \frac{K+N}{N} \ll \frac{A}{N}$. $\frac{A}{N} \cap \frac{K+N}{N} = \frac{A\cap(K+N)}{N} = \frac{(A\cap K)+N}{N}$ (by modular law). Let $\frac{(A\cap K)+N}{N} + \frac{L}{N} = \frac{A}{N}$, where $L \subseteq A$ and $N \subseteq L$ then $\frac{(A\cap K)+N+L}{N} = \frac{A}{N}$, hence $(A \cap K) + N + L = A$, but $N \subseteq L$, then $(A \cap K) + L = A$, and $A \cap K \ll A$, then $L = A$ and hence $\frac{L}{N} = \frac{A}{N}$, therefore $\frac{A}{N} \cap \frac{K+N}{N} \ll \frac{A}{N}$

2- COFINITELY SUPPLEMENTED MODULE

In this section, we recall the concept of cofinitely supplemented modules with some properties see [6].

Definition (2.2.1):

A module M is called cofinitely supplemented module (for short cof-supplemented) if for every cofinite sub module L of M. There exists a submodule N of M such that M = L + N and N ∩ L≪N.

Remark (2.2.2):

It is clear that every supplemented module is cof- supplemented. The converse in general is not true, concider the following example. Q as Z-module is cof-supplemented module. But it is known That Q is not supplemental.

Proposition (2.2.3):

Let M be a finitely generated R-module. Then M is supplemented module if and only if M is cof-supplemenited.

Proof:

To show that M is supplemented module. Let L be a submodule of M since M is afinitely generated R-module. Then $\frac{M}{L}$ is a finitely generated hence L is acofinite submodule of M. But M is cof-supplemented therefore L has supplemented in M thus M is supplemented module the converse is clear.

Proposition (2.2.4):

Let M be a cof-supplemented. Let B be a submodule of M then $\frac{M}{B}$ is a cof-supplemented.

Proof:

Let B be a submodule of M and let $\frac{K}{B}$ be any cofinite submodule of $\frac{M}{B}$ such that $\frac{M}{K} \cong \frac{\frac{M}{B}}{\frac{K}{B}}$ is finitely generated. Then K is a cofinite submodule of M since M is a cof-supplemented. Then there exists a submodule C of M such that M=K+C K ∩ ≪C. Now $\frac{M}{B} = \frac{K+C}{B} = \frac{K}{B} + \frac{C+B}{B}$ to show $\frac{K}{B} \cap \frac{C+B}{B} \ll \frac{C+B}{B}$. Let $(\frac{K}{B} \cap \frac{C+B}{B}) + \frac{V}{B} = \frac{C+B}{B}$

with $(\frac{C+B}{B}) = \frac{C+B}{V}, \frac{K\cap(C+B)}{B} = \frac{B+(K\cap C)}{B}$ then

$\frac{B+(K\cap C)}{B} + \frac{V}{B} = \frac{C+B}{B}$ and B+(K∩C) +V=C+B and $(\frac{C+B}{V}) = \frac{C+B}{V}$, but $K \cap C \ll C \leq C+B$ and $K \cap C \ll C+B$ thus

V=C+B and $\frac{V}{B} = \frac{C+B}{B}$, there fore $\frac{M}{B}$ is a cof- supplementel module.

Proposition (2.2.5):

Let $M=M_1 \oplus M_2$, then M_1 and M_2 are a cof-supplemented module if and only if M is a cof-supplemen ted module.

Proof:

Let L be a cofinite sub module of M. Then $M=L+M_1+M_2$ now $\frac{M2}{M2\cap(L+M1)} \cong \frac{M2+LM1}{L+M1} = \frac{M}{L+M1} \cong \frac{\frac{M}{L}}{\frac{L+M1}{L}}$

which is finitely generated. Hence $M_2 \cap(L+M_1)$ is a cufinite submodule of M_2. Since M_2 is cof-supplemented. Then there exists asubmodule H of M_2

such that $M_2 = H + [M_2 \cap (L + M_1)]$ with $H \cap (L + M_1) \ll H$ we have $M = L + M_1 + M_2 = L + M_1 + M_2 \cap (L + M_1) + H = M_1 + L + H$ and since $M_1 \cap (L + H)$ is a cufinite submodule of M_1 and M_1 is a cof-supplemented. Then there exists a submodule G of M_1 such that

$M_1 = G + [M_1 \cap (L + H)]$ and $G \cap (L + H) \ll H$, hence

$M = G + M_1 \cap (L + H) + L + H = L + H + G$ and $(H + G) \cap L \leq [H \cap (L + M_1)] + [G \cap (L + H)] \ll H + G$

there fare M is a cof-supplemented module.

Conversely $M_2 \cong \frac{M}{M1}$ and M is a cof-supplemented module by proposition (2.1.6) $\frac{M}{M1}$ is

A cof-supplemented module. Then M_2 is a cof-supplemented module. similarity M_1 is a cof-supplemented module.

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