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On Stable Submodules

A Thesis Submitted to the council of college of Education/ University of Misan in Partial Fulfillment of the Requirements for the Degree of Bachelor in mathematics.

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ وَإِنَّهُ فِي أُمِّ الْكِتَابِ لَدَيْنَا لَعَلِيٌّ حَكِيمٌ ﴾

صدق الله العلي العظيم

سورة الزخرف :الآيه 4

الإهداء

إلى من كانوا مشاعل النور، وأئمة الهدى، وسفن النجاة...

إلى سادتنا وقادتنا، الأئمة الأربعة عشر، الذين أناروا لنا طريق الحق، وساروا بنا نحو دروب المعرفة والإيمان...

إلى رسول الله محمد (صلى الله عليه وآله)،

والى الأمير علي بن أبي طالب عليه السلام ،

وإلى بضعته الطاهرة فاطمة الزهراء (عليها السلام)،

وإلى الأئمة المعصومين من ذريته الطيبة (عليهم السلام)، الذين بذلوا أرواحهم في سبيل إعلاء كلمة الله ونشر العلم والحكمة...

الباحث

الشكر والتقدير

في البداية ، الشكرُ لله والحمدُ لله ، جلّ في علاه، فأليه ينسب الفضل كله في إكمال هذا العمل.

وبعد الحمدُ لله , فإننا نتوجه إلى دكتورتنا ومشرفتنا على البحث

(الدكتورة هبة ربيع)

بالشكر والتقدير التي لن تفي إي كلمات في حقها، فلولا مِتابعتها ودعمها المستمر لما تم هذا العمل،

وبعدها فالشكر موصول لكل أساتذتنا الذين تتلمذنا على أيديهم في كل مراحلنا

الدراسية حتى نتشرف بوقوفنا أمام حضرتهم اليوم.

Supervisor approval

I certify that this research (On Stable Submodules) submitted by the students (Hassan Yasen Taher) took place under my supervision at the University of Misan/ College of Education/ Department of Mathematics. It is part of the requirements for obtaining a bachelor's degree in the College of Education / Department of Mathematics.

Supervisor

Signature:

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Date: / / 2025

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ABSRACT

Let M be a right module with identity over a commutative ring R with identity. This work studies a class of submodules called stable, with some examples and properties.

INRODUCTION

Throughout, all rings are associative with identity, and modules are rings unitary. In this work, we will study the concepts of the pure submodule and the purely small submodule.

This research has two chapters:

In chapter one, we recall some properties about the module.

In chapter two, study the stable submodule with examples and properties.

CHAPTER ONE

(BACKGROUND on MODULES)

Chapter One : Background on Modules

In this chapter we will recall the definition of modules with some examples and properties. For more details, see[4].

Definition (1.1): A ring is an ordered triple $(R, +, \cdot)$ consisting a nonempty set R with two binary operations $+$ and $\cdot$ defined on R such that

1) $(R, +)$ is a commutative group,
2) $(R, \cdot)$ is a semigroup, and
3) The operation $\cdot$ is distributive over the operation $+$.

Definition (1.2):

1) A commutative ring is a ring $(R, +, \cdot)$ in which multiplication is a commutative operation, $a \cdot b = b \cdot a$ for all $a, b \in R$.
2) A ring with identity is a ring $(R, +, \cdot)$ in which there exists an identity element for the operation of multiplication, normally represented by the symbol 1, so that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$.

Example (1.3): The mathematical system $(\mathbb{Z}, +, \cdot)$ is a a commutative ring with identity 1 , since

1) $(\mathbb{Z}, +)$ is a commutative group ,
2) $(\mathbb{Z}, \cdot)$ is a semigroup, and
3) $\forall a, b, c \in \mathbb{Z} \quad a . (b + c) = (a . b) + (a . c)$

And $(b + c) . a = (b . a) + (c . a)$

4) The multiplication is a commutative operation with identity.

Definition (1.4):

Let $(R\ ,+,\cdot)$ be a ring and let $\emptyset \neq\ I \subseteq R$, then $(I\ ,+,\cdot)$ is called an ideal of $(R\ ,+,\cdot)$ if and only if

1) $a - b \in\ I, \forall a, b \in I$,
2) $\forall r \in R$ and $\forall\ a \in I$ then $ar\ \in I$ and $ra\ \in I$.

- If $ar \in I$ for all $r\ \in R$, $a \in I$ then I is called right ideal.
- If $ra\ \in I$ for all $r \in R$, $a\ \in I$ then I is called left ideal.
- If I is both right and left ideal then I is called two sided ideal or simply ideal.

Example(1.5): The set of all even integer number $(\mathbb{Z}_e\ ,+,\cdot)$ is an ideal of the ring $(\mathbb{Z}\ ,+,\cdot)$.

Definition(1.6) : Let $(R,+,\cdot\ )$ be a ring and $(I,+,\cdot\ )$ be an ideal of R. Then the set $\frac{R}{I} = \{r + I | r \in R\}$ with the following operation

$$(a + I) + (b + I) = (a + b) + I$$

and

$$(a + I) \cdot (b + I) = ab + I$$

Be a ring which called quotient ring.

Definition(1.7) :

A non zero ideal $(I,+,\cdot)$ of a ring $(R,+,\cdot)$ is called Maximal ideal. If $I \neq R$ and there exists no proper ideal of a ring R containing I, i.e. I is a maximal ideal of R if $I \neq R$, and if there exists an ideal $(J,+,.)$ in R with $I \subset J \subseteq R$, then $J = R$.

Example (1.8): In the ring $(\mathbb{Z}_{12}, +_{12}, ._{12})$ the following ideals are proper

1) $\mathrm{I}_1 = \langle\bar{2}\rangle = (\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\},+_{12}, ._{12})$.
2) $\mathrm{I}_2 = \langle\bar{3}\rangle = (\{\bar{0},\bar{3},\bar{6},\bar{9}\},+12, ._{12})$.
3) $\mathrm{I}_3 = \langle\bar{4}\rangle = (\{\bar{0},\bar{4},\bar{8}\},+_{12}, ._{12})$.
4) $\mathrm{I}_4 = \langle\bar{6}\rangle = (\{\bar{0},\bar{6}\},+_{12}, ._{12})$

I_1 and I_2 are maximal ideals in a ring $\mathbb{Z}_{12}$, since there is no a proper ideals of a ring $\mathbb{Z}_{12}$ containing $\mathrm{I}_1$and I_2.

But I_3 is not maximal ideal in $\mathbb{Z}_{12}$, since $\langle\bar{4}\rangle \subset \langle\bar{2}\rangle$ and $\mathrm{I}_4 =$ is not maximal ideal in $\mathbb{Z}_{12}$, since $\langle\bar{6}\rangle \subset \langle\bar{2}\rangle$.

Definition (1.9): A ring $(F,+,.)$ is said to be field if and only if $(F,+,.)$ is a commutative ring with identity and every a non zero element $a \in F$ is invertible element**.**

Example (1.10): The rings $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are filed.

<u>Definition(1.11)</u>: Let R be a ring. A right R-module M is

(I) an additive abelian group M together with

(II) a mapping

$$M \times R \to M \;\; with \; (m, r) \to mr.$$

Called module multiplication , for which we have

1- Associative law: $(mr_1)r_2 = m(r_1r_2)$.
2- Distributive laws: $(m_1 + m_2)r = m_1r + m_2r,\; m(r_1 + r_2) = mr_1 + mr_2$.
3- Unitary law: $m1 = m$
   For each $m, m_1 \; and \; m_2 \in M$ and $r, r_1 \; and \; r_2 \in R$.

<u>Example (1.12):</u> The set of real numbers $\mathbb{R}$ is a $\mathbb{Q}$-module.

<u>Solution:</u>

$(i)(\mathbb{R}, +)$ is an abelian group ,

$(ii)(\mathbb{Q}, +, \cdot)$ be a ring with identity 1,

Then, there is a mapping $f\colon \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$, such that $f\big((r, m)\big) = r.m$, $\forall (r, m) \in \mathbb{Q} \times \mathbb{R}$ and f satisfy the following condition: $\forall\, r, r_1, r_2 \in \mathbb{Q} \; and \; \forall\, m, m_1, m_2 \in \mathbb{R}$ then

1- $f\big((r_1 + r_2, m)\big) = (r_1 + r_2).m = r_1.m + r_2.m$
2- $f\;((\mathrm{r}, m_1 + m_2\;)) = r.(m_1 + m_2) = r.m_1 + r.m_2$.
3- $f\big((r_1.r_2, m)\big) = (r_1.r_2).m = r_1.(r_2.m)$
4- If 1 is the identity element of a ring $\mathbb{Q}$, then $1.m = m, \forall m \in \mathbb{R}$.

Definition (1.13): Let M be a right R-module . A subset A of M is called a submodule of M , notationally $A \leq M$ (or also $A_R \leq M_R$) if A is a right R-module with respect to the restriction of the addition and module multiplication of M to A.

Lemma (1.14): Let M be a right R-module . If A is a subset of M and

$\boldsymbol{A} \neq \emptyset$ then the following are equivalent:

1) $A \leq M$.
2) A is a subgroup of the additive group of M and for all $a \in A$ and all $r \in R$ we have that $ar \in A$ (where ar is the module multiplication in $\boldsymbol{M}$).
3) For all $a_1, a_2 \in A$, $a_1 + a_2 \in A$(with respect to addition in $\boldsymbol{M}$) and for all $a \in A$ and all $r \in R$ we have $ar \in A$.

Example(1.15): $\mathbb{Q}$ is submodule of $\mathbb{R}$ as a $\mathbb{Z}$-module since $\emptyset \neq \mathbb{Q} \subseteq \mathbb{R}$ and

1) $a + b \in \mathbb{Q}, \forall\, a, b \in \mathbb{Q}$.
2) $r.a \in \mathbb{Q}\,, \forall\, r \in \mathbb{Z}\; and\; \forall\, a \in \mathbb{Q}$.

Definition (1.16): Let $\Lambda = \{A_1, \ldots\ldots, A_n\}$ be a set of submodules $A_i \leq M$,

then every element from $\sum_{j=1}^{n} A_j$ can be wtitten in the from

$$\sum_{j \in 1}^{n} a_j \quad with\; a_j \in A_i$$

the missing summands a_j can be added as $a_j = 0$. Generally it should be emphasized that the representation $\sum_{i\in I} a_i$ of the elements of the sum neednot be unique.

Lemma (1.17): Let Γ be a set of submodules of a module M , then

$$\bigcap_{A\in\Gamma} A \ := \ \{m \ \in \ A \mid \forall\, A \in \Gamma\}$$

is a submodule of M.

Definition (1.18): M is called the internal direct sum of the set $\{B_i | i \in I\}$ of submodules $B_i \leq M$, in symbols:

$$M = \bigoplus_{i\in I} B_i : \Leftrightarrow \begin{cases} (1)\ M = \sum_{i\in I} B_i \ \wedge \\ (2)\ \forall j \in I[B_j \cap \sum_{\substack{i\in I \\ i\neq j}} B_i = 0]\cdot \end{cases}$$

$M = \bigoplus_{i\in I} B_i$ is also said to be a direct decomposition of M into the sum of the submodules $\{B_i | i \in I\}$.

In the case of a finite index set , say $I = \{1, \dots, n\}$ M is also written as $M = B_1 \oplus \dots \oplus B_n$.

Definition (1.19): Let $\Gamma = \{M_i\}_{i\in I}$, where M_i is an R-module for each $i \in I$. then the direct sum of all M_i is

$$\bigoplus_{i\in I} M_i : \Leftrightarrow \begin{cases} (1)\ \sum_{i\in I} M_i \ \wedge \\ (2)\ \forall j \in I[M_j \cap \sum_{\substack{i\in I \\ i\neq j}} M_i = 0]\cdot \end{cases}$$

Definition (1.20): A submodule $B \leq M$ is called a direct summand of M if and only if there exists $C \leq M$ such that $M = B \oplus C$.

Example(1.21): In the $\mathbb{Z}$-module $\mathbb{Z}_6$, the submodule $\langle\bar{2}\rangle = \{\bar{0}, \bar{2}, \bar{4}\}$ is a direct summand of $\mathbb{Z}_6$, since there is a submodule $\langle\bar{3}\rangle = \{\bar{0}, \bar{3}\}$. Such that $\langle\bar{2}\rangle + \langle\bar{3}\rangle = \mathbb{Z}_6$ and $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \{\bar{0}\}$.

Definition (1.22): Let M be an R-module and let $m_0 \in M$. Then $\langle m_0 \rangle$ $m_0R = \{m_0 r | r \in R\}$ is a submodule of M which is called the cyclic submodule of M generated by m_0. A module M is called cyclic if there is $m_0 \in M$ such that $M = m_0R$.

Example(1.23): All the submodules of the $\mathbb{Z}$-module $\mathbb{Z}_6$ are cyclic,

$\mathbb{Z}_6 = \langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, $\langle \bar{0} \rangle = \{\bar{0}\}$, $\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}\}$ and $\langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$.

Definition (1.24): A module $M \neq 0$ is called directly indecomposable if and only if 0 and M are the only direct summands of M.

Example(1.25): In the $\mathbb{Z}$-module $\mathbb{Z}_4$, the all submodules are: $\langle \bar{0} \rangle$, $\langle \bar{2} \rangle$ and $\mathbb{Z}_4$. The only direct summands of $\mathbb{Z}_4$ are: $\langle \bar{0} \rangle$ and $\mathbb{Z}_4$. So $\mathbb{Z}_4$ is indecomposable module.

Lemma (1.26): (modular law) Let $A, B, C \leq M$ and $B \leq C$. Then $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Definition(1.27): Let A and B be both right R-module. A homomorphism $f: A \longrightarrow B$ is a mapping which satisfies:

$\forall a_1, a_2 \in A \; \forall \, r_1, r_2 \in R, f(a_1 r_1 + a_2 r_2) = f(a_1) r_1 + f(a_2) r_2$.

Definition (1.28): A homomorphism $f: A_R \longrightarrow B_R$ is called

endomorphism of A if and only if $A = B$;

a monomorphism if and only if f is one to one;

an epimorphism if and only if f is onto;

an isomorphism if and only if f is onto and one to one.

Definition (1.29): Let $f: A_R \longrightarrow B_R$ be a homomorphism with $U \leq A$, $V \leq B$. Then $f(U): \{f(u) | u \in U\} \leq B$ and $f^{-1}(V) = \{a \in A | f(a) \in V\} \leq A$.

Definition (1.30) : Let $f: A_R \longrightarrow B_R$ be a homomorphism. Then $Ker\, f = f^{-1}(0) = \{\, a \in A | f(a) = 0\}$.

Example(1.31): Let $\mathbb{Z}$ be an $2\mathbb{Z}$-module a function $f: \mathbb{Z} \to \mathbb{Z}$ where $f(a) = 3a$ for each $a \in \mathbb{Z}$ is an $2\mathbb{Z}$-module homomorphism, let $.r \in 2\mathbb{Z}$ then we get

1- $f\ (a + b) = 3(a + b) = 3a + 3b =$ f$(a) +$ f(b)
2- $f\ (ar) = 3(ar) = (3a)r =$ f(a)r
   kernel (f) = {0} , Im (f) =3$\mathbb{Z}$

Also, we can show that f is monomorphism

let $a, b \in \mathbb{Z}$ where f(a)=f(b) then $3a = 3b$ consequently $a = b$. so f is injective .

Definition (1.32)): A submodule A of a module M is called small (superfluous) in M if whenever $\mathrm{A + U = M}$ then $\mathrm{U = M}$ for each $\mathrm{U \leq M}$. Which denoted by ($A \ll M$).

Example (1.33): In $\mathbb{Z}_4$ as $\mathbb{Z}$-module, the all submodules are: $\mathbb{Z}_4 = \langle\bar{1}\rangle = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$, $\langle\bar{0}\rangle = \bar{0}\mathbb{Z}_6 = \{\bar{0}\}$ and $\langle\bar{2}\rangle = \bar{2}\,\mathbb{Z} = \{\bar{0},\bar{1}\}$. $\langle\bar{2}\rangle$ is small in $\mathbb{Z}_4$, Since the only case that we have is $\langle\bar{2}\rangle + \mathbb{Z}_4 = \mathbb{Z}_4$.

Definition (1.34) : A submodule K of an R-module $M$is called essential if K has non zero intersection with every non zero submodule . Denoted by $(K \leq_e M)$.

I.e. If $K \cap A \neq 0 \quad \forall\, A \leq M \;, A \neq 0$, then $K \leq_e M$.

Example (1.35): In $\mathbb{Z}_4$ as $\mathbb{Z}$-module, the all submodules are: $\mathbb{Z}_4$=$\{\bar{0},\bar{1},\bar{2},\bar{3}\}$,$\langle\bar{0}\rangle = \{\bar{0}\}$ and $\langle\bar{2}\rangle = \{\bar{0},\bar{2}\}$.$\langle\bar{2}\rangle$ is essential submodule of $\mathbb{Z}_4$ since $\langle\bar{2}\rangle \cap \mathbb{Z}_4 = \langle\bar{2}\rangle$.

Definition (1.36): The Jacobson radical of a module M is the sum of all small submodule of M. Denoted by $Rad(M)$**.**

$$Rad(M) = \sum\{L \leq M | L \text{ } is \text{ } small \text{ } in \text{ } M\,\}$$

CHAPTER TWO

STABLE SUBMODULES

Chapter Two : Stable Submodules

In this chapter, we will recall the concept of the stable submodules with some properties. See [1].

Definition(2.1): Let M be an R-module. A submodule $N$of M is said to be stable. If $f(N) \subseteq N$ for each R-homomorphism $f: N \to M$.

Example and Remark(2.2):

1) The trivial submodules are always stable, since for each module M and each R-homomorphism $f: M \to M$, $f(M) \subseteq M$. Also for the zero submodule $\{0\}$ for each R-homomorphism $f: \{0\} \to M$, $f(\{0\}) = \{0\} \subseteq \{0\}$.
2) Every module M has at least two stable submodules are: $\{0\}$ and M.
3) In $\mathbb{Z}$ as $\mathbb{Z}$-module, the submodule $\langle 2 \rangle = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$is not stable since there exists a homomorphism $f: \langle 2 \rangle \to \mathbb{Z}$, which define by $f(2x) = 3x$ for each $x \in \mathbb{Z}$, f is well defined since if $x_1, x_2 \in \mathbb{Z}$ such that $2x_1 = 2x_2$. So $x_1 = x_2$ and $3x_1 = 3x_2$. Hence, $f(2x_1) = f(2x_2)$. Also, f is a homomorphism. f$(2x_1 + 2x_2) = f\big(2(x_1 + x_2)\big) = 3(x_1 + x_2) = 3x_1 + 3x_2 = f(x_1) + f(x_2)$. And $f((2x)r) = f(2(xr)) = 3(xr) = f(x) \cdot r$, for each $x_1, x_2, r \in \mathbb{Z}$, but $f(2\mathbb{Z}) \nsubseteq 2\mathbb{Z}$ since $f(2) = 3 \notin 2\mathbb{Z}$.
4) The direct sum of stable submodules need not to be stable. For example the ring $\mathbb{R}$ as an $\mathbb{R}$-module and $\{0\}$ are stable submodules but $\{0\} \oplus \mathbb{R}$ is not stable submodule in $\mathbb{R} \oplus \mathbb{R}$. Since there exists an

$\mathbb{R}$-homomorphism f: $\{0\}\oplus \mathbb{R} \to \mathbb{R}\oplus \mathbb{R}$ which define by $f\big((0,r)\big) = (r,0)$. So $f\big((0,1)\big) = (1,0) \notin \{0\}\oplus \mathbb{R}$. Hence, f $(\{0\}\oplus \mathbb{R}) \nsubseteq \{0\}\oplus \mathbb{R}$.

Now, we will study the properties of the stable submodule.

Proposition (2.3):

The sum of any family of stable submodules is stable.

Proof: Let $\{A_1, A_2, \dots, A_n\}$ be a family of stable submodules of an R-module M and let $f\colon \sum_{i=1}^{n} A_i \to M$ be an R-homomorphism. We know that $\sum_{i=1}^{n} A_i = \{a_1 + a_2 + \cdots + a_n | a_1 \in A_1, \dots, a_n \in A_n\}$. Let $x \in f(\sum_{i=1}^{n} A_i)$. Then $x = f(a_1 + a_2 + \cdots + a_n)$ for some $a_1 \in A_1, \dots, a_n \in A_n$, since f is homomorphism $x = f(a_1 + a_2 + \cdots + a_n) = f(a_1) + f(a_2) + \cdots + f(a_n)$. There exists the inclusion homomorphism for each A_i (i=1,2,..,n), since $A_i \subseteq \sum_{i=1}^{n} A_i$, $i_1\colon A_1 \to \sum_{i=1}^{n} A_i$, $i_2\colon A_2 \to \sum_{i=1}^{n} A_i$, …, $i_n\colon A_n \to \sum_{i=1}^{n} A_i$ which define as: $i_1(a_1) = a_1, i_2(a_2) = a_2, .., i_n(a_n) = a_n$.Then there exists a homomorphisms $i_1 \circ f\colon A_1 \to M$, $i_2 \circ f\colon A_2 \to M$, …, $i_n \circ f\colon A_n \to M$; $A_1, A_2, \dots, A_n$ are stable submodules. So $i_1 \circ f\ (A_1) \subseteq A_1$ with $i_1 \circ f\ (a_1) = f\ (a_1) \in A_1, i_2 \circ f\ (A_2) \subseteq A_2$ with $i_2 \circ f\ (a_2) = f\ (a_2) \in A_2, \dots,$ $i_n \circ f\ (a_n) = f\ (a_n) \in A_n$. Hence $f(a_1) + f(a_2) + \cdots + f(a_n) \in \sum_{i=1}^{n} A_i$ and $x \in \sum_{i=1}^{n} A_i$. Therefore, $f\ (\sum_{i=1}^{n} A_i) \subseteq \sum_{i=1}^{n} A_i$ and $\sum_{i=1}^{n} A_i$ is an stable submodule.

Proposition (2.4): The Jacobson radical of any module is a stable submodule**.**

Proof: Let M be an R-module. For any R- homomorphism $f:\mathcal{J}(M) \to M$, $\mathcal{J}(M) = \sum_i B_i$, where B_i are the small submodules of M. Then $f\big(\mathcal{J}(M)\big) = f(\sum_i B_i) = \sum_i f(B_i)$, by lemma (1.9) in [1] (homomorphic image of small submodule is small) and the sum of the small submodules is small. So $f(B_i)$ is small submodule of M ,$\forall i$ and $\sum_i f(B_i)$ is small submodule of M. Hence, $f\big(\mathcal{J}(M)\big) = \sum_i f(B_i) \subseteq \mathcal{J}(M)$. Therefore, $\mathcal{J}(M)$ is a stable submodule of M.

Goldie[2] and Wong[3], defined the closure submodule. Let N be a submodule of a module M.

Then $cl(N) = \{m \in M | mI \subseteq N \text{ for some essential ideal } I \ of\ R\}$. Clearly, $N \subseteq cl(N)$.

Proposition (2.5): If N is a stable submodule of the module M , then $cl(N)$ is a stable submodule of M.

Proof: For each R-homomorphism $f: cl(N) \to M$ and $y \in f\big(cl(N)\big)$. then there exists $m \in cl(N)$ such that $y = f(m)$, then $mI \subseteq N$ for some essential ideal I of R. *Hence* $f(m)I = f(mI) \subseteq f(N)$. *Since* $N \subseteq cl(N)$ *and* N *is stable then* $f(N) \subseteq N$. *So* $f(m)I \subseteq N$. *Thus,* $f(m) \in cl(N)$ *and* $f\big(cl(N)\big) \subseteq cl(N)$.

References

[1] **M.S. Abbas**, On fully stable modules, Ph. D. thesis, Univ. of Baghdad, 1990.

[2] **A.W. Goldie**, Torsion-free modules and rings, J. Algebra1 (1964): 268-287.

[3] **R.E. Johnson**, E.T. Wong: Quasi-injective modules and irreducible rings, J. London Math. Soc., 39(1961): 260-268.

[4] **F. Kasch**: Modules and rings, Academic Press, London, 1982.