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# Homotopy perturbation method for solving time-fractional nonlinear Variable-Order Delay Partial Differential Equations

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ABSTRACT

The homotopy perturbation method is extend to derive the approximate solution of the variable order fractional partial differential equations with time delay. The variable order fractional derivative is taken in the Caputo sense. An approximation formula of the Caputo derivative of fractional variable order is presented in terms of standard (integer order) derivatives only. Then the original problem will be transformed into a systems of partial differential equations with delay. By employing the homotopy perturbation method the explicit approximate solutions are found. The error and convergence analysis of the homotopy perturbation method has been discussed for the applicability of the method. The absolute errors and the approximate solutions are presented graphically and by tables at the values of various variable fractional order. From the results of the illustrated examples, we can Judge that the homotopy perturbation method is very effective, and simple accelerates the rapid convergence of the solution.

## 1. Introduction

The number of scientific and engineering problems involving fractional calculus (FC) is already very large and still growing. One of the main advantages of the fractional calculus is that the fractional derivatives provide an excellent approach for the description of memory and hereditary properties of various materials and processes.[1–7] Many of numerical methods for solving fractional order ordinary and partial differential equations have been proposed see for instance.[8–18] Researchers have found that many dynamic processes exhibit fractional order behavior that may vary with time or space which indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems.[1] With the variable order calculus the non-local properties are more evident and numerous applications have been found in physics, control and signal processing.[19,20] In 1993, Samko and Ross devoted themselves to investigate operators when the order $\alpha$ is not constant during the process, but variable on time.[21–23] Lorenzo and Hartley[24,25] suggested the concept of a variable operator is allowed to vary either as a function of the independent variable of integration or differentiation $(t)$, or as a function of some other variable $(x)$, they also explored more deeply the concept of variable integration and differentiation and sought the relationship between the mathematical concepts and physical processes. In the field of (FC), the study of fractional order partial differential equations (FPDEs) has particularly been attacked by many authors.

FPDEs are a fascinating subject they are frequently used to explain a variety of phenomena in real-world situations including signal processing, control theory, fluid flow,potential theory, information theory, finance and entropy.[26–29] In fact, variable order fractional partial differential equations (VFPDEs) is a generalization of the (FPDEs) and it is appear in many applications of physics and engineering.[29–31] It is well known that in real world problems delay is important to model certain processes and dynamical systems.[32,33] However, there are still few works in the literature dedicated to (VFPDEs)[29–31] up to the our knowledge there has been no works on VFPDEs with proportional delays. Therefore this encouraged us in this paper to handle and finding the approximate solutions of the VFPDEs with proportional delays using HPM with the aid of approximating the variable order fractional derivative using the approach given in Ref. 34. HPM is a semi-analytic method for finding the approximate solutions of non-linear problems. It was suggested by He,[35,36] the homotopy idea in topology is combined with conventional perturbation approach to crate HPM. Without linearization or discretization, (HPM) can provide both approximate and exact solutions. The application of the (HPM) has appeared in many papers actually during the recent years, which shows that the method is powerful technique for studying the approximate solutions.[35–44] The following time VFPDEs with proportional delays are taken into consideration in this study

$$ {}_{0}^{C}D_{\varrho}^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) = F\left(\vartheta,\varrho, v(p_0\vartheta, q_0\varrho), \frac{\partial}{\partial\vartheta} v(p_1\vartheta, q_1\varrho), \dots, \frac{\partial^n}{\partial\vartheta^n} v(p_n\vartheta, q_n\varrho)\right), \quad (1.1) $$

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with the initial condition,

$$v^m(\vartheta, 0) = g_m(\vartheta), \tag{1.2}$$

where $p_i, q_j \in (0,1)$ for $(i,j) \in N, g_m(\vartheta), m = 0,1,2,\ldots$ are given initial values and $F$ is the partial differential operator.

## 2. Preliminaries

**Definition 2.1.** The Riemann–Liouville variable-order fractional integral operator with order $n-1 < \delta(\vartheta,\varrho) \le n,\ \varrho > 0$ of $v(\vartheta,\varrho)$ is defined as[45]:

$$I_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) = \frac{1}{\Gamma(\delta(\vartheta,\varrho))} \int_0^\varrho (\varrho-\rho)^{\delta(\vartheta,\varrho)-1}\, v(\vartheta,\rho)\, d\rho, \tag{2.1}$$

where $\varrho > 0$ and $\Gamma(.)$ is the Gamma function.

According to Definition 2.1, variable-order fractional integration satisfy the following property:

$$I_\varrho^{\delta(\vartheta,\varrho)}\ \varrho^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\delta(\vartheta,\varrho)+1)} \varrho^{\beta+\delta(\vartheta,\varrho)} & ,\beta > -1 \\ 0, & otherwise. \end{cases} \tag{2.2}$$

**Definition 2.2.** In the Caputo experience, the variable-order fractional derivative operator of $v(\vartheta,\varrho)$ is given as follows[46]:

$$\begin{aligned} {}_0^C D_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) &= I_\varrho^{n-\delta(\vartheta,\varrho)}\ D_\varrho^n\ v(\vartheta,\varrho) \\ &= \frac{1}{\Gamma(n-\delta(\vartheta,\varrho))} \int_0^\varrho (\varrho-\rho)^{n-\delta(\vartheta,\varrho)-1} \frac{\partial^n v(\vartheta,\rho)}{\partial\rho^n}\, d\rho \end{aligned} \tag{2.3}$$

for $n-1 < \delta(\vartheta,\varrho) \le n,\quad \varrho > 0,\quad$ and $n \in Z^+$, based on Eq. (2.3), we find the following relation:

$${}_0^C D_\varrho^{\delta(\vartheta,\varrho)}\ \varrho^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-\delta(\vartheta,\varrho)+1)} \varrho^{m-\delta(\vartheta,\varrho)} & ,n \le m \in N \\ 0, & otherwise. \end{cases} \tag{2.4}$$

## 3. Approximation of variable order Caputo derivative

In this section, the variable order Caputo fractional derivative will be approximated using the proposed method in Ref. 34 , which is given in the following formula:

$${}_0^c D_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) \simeq A\varrho^{1-\delta(\vartheta,\varrho)} \frac{\partial v}{\partial\varrho}(\vartheta,\varrho) + \Sigma_{\rho=1}^N B_\rho \varrho^{1-\rho-\delta(\vartheta,\varrho)} U_\rho(\vartheta,\varrho), \tag{3.1}$$

with

$$A = \frac{1}{\Gamma(2-\delta(\vartheta,\varrho))}[1 + \sum_{\ell=1}^N \frac{\Gamma(\delta(\vartheta,\varrho)) - 1 + \ell}{\Gamma(\delta(\vartheta,\varrho))\ell!}],$$

$$B_\rho = \frac{\Gamma(\delta(\vartheta,\varrho) - 1 + \rho)}{\Gamma(1-\delta(\vartheta,\varrho))\Gamma(\delta(\vartheta,\varrho))(\rho-1)!},$$

$$U_\rho(\vartheta,\varrho) = \int_0^\varrho s^{\rho-1} \frac{\partial v(\vartheta,s)}{\partial\varrho}(\vartheta,s)ds.$$

It is clear that, as $N$ increases, the error of the approximation decreases and the given approximation formula converges to the fractional derivative.

## 4. Application of (HPM) for solving (VFPDDEs)

Consider the following VFPDDEs:

$${}_0^C D_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) = F\left(\vartheta, \varrho, v(p_0\vartheta, q_0\varrho), \frac{\partial}{\partial\vartheta} v(p_1\vartheta, q_1\varrho), \ldots, \frac{\partial^n}{\partial\vartheta^n} v(p_n\vartheta, q_n\varrho)\right), \tag{4.1}$$

where, $\vartheta, \varrho \in [0,1]$ and ${}_0^C D_\varrho^{\delta(\vartheta,\varrho)}$ is the fractional derivative of order $\delta(\vartheta,\varrho)$ with respect to $\varrho, m-1 < \delta(\vartheta,\varrho) \le m, m \in N^+, u^m(\vartheta,0) = g_m(\vartheta), m = 0,1,2,\ldots,p_i,q_j \in (0,1)$, for $i,j \in N$, and $F$ the partial differential operator.

At the beginning of our method, we will approximate the variable order fractional derivative by substituting Eq. (3.1) into Eq. (1.1), and thus the original problem (1.1)–(1.2) will turn into a system of partial differential equations in the following form:

$$A\varrho^{1-\delta(\vartheta,\varrho)} \frac{\partial v}{\partial\varrho}(\vartheta,\varrho) + \sum_{\rho=1}^N B_\rho \varrho^{1-\rho-\delta(\vartheta,\varrho)} U_\rho(\vartheta,\varrho) = F(\vartheta,\varrho, v(p_0\vartheta, q_0\varrho), \frac{\partial v}{\partial\vartheta}(p_1\vartheta, q_1\varrho), \ldots), \tag{4.2}$$

and

$$U_k(\vartheta,\varrho) = \int_0^\varrho s^{k-1} \frac{\partial v(\vartheta,s)}{\partial\varrho}(\vartheta,s)ds \quad ,k = 1,2,3,\ldots \quad .$$

Subject to the following boundary conditions:

$$v^m(\vartheta,\varrho) = g_m(\vartheta)$$

$$U_k(\vartheta,0) = 0, \qquad k = 1,2,3,\ldots,N.$$

If $b(\vartheta,\varrho) = A\varrho^{1-\delta(\vartheta,\varrho)}$, we get

$$b(\vartheta,\varrho)\frac{\partial v}{\partial\varrho} + \sum_{k=1}^N B_k \varrho^{1-k-\delta(\vartheta,\varrho)} U_k(\vartheta,\varrho) = F(\vartheta,\varrho, v(p_0\vartheta, q_0\varrho), \frac{\partial v}{\partial\vartheta}(p_1\vartheta, q_1\varrho), \ldots), \tag{4.3}$$

and

$$U_k(\vartheta,\varrho) = \int_0^\varrho s^{k-1} \frac{\partial v(\vartheta,\varrho)}{\partial s} ds, \qquad k = 1,2,\ldots,N, \tag{4.4}$$

Now, define $Lv(\vartheta,\varrho)$ by

$$Lv(\vartheta,\varrho) = b(\vartheta,\varrho)\frac{\partial v}{\partial\varrho}. \tag{4.5}$$

Then according to the homotopy perturbation theory,[35,36] we can construct the following homotopy for Eqs. (4.3)–(4.4):

$$Lv - Lv_0 + H\ [Lv_0 + \sum_{k=1}^N \varrho^{1-k-\delta(\vartheta,\varrho)} U_k(\vartheta,\varrho) - F(\vartheta,\varrho, v(p_0\vartheta, q_0\varrho), \ldots)] = 0, \tag{4.6}$$

and

$$U_k - U_{k_0} + H[U_{k_0} - \int_0^t s^{k-1} \frac{\partial v}{\partial\varrho} ds] = 0. \tag{4.7}$$

If the embedded parameter $H$ based on the concept of the classic perturbation method, so the solution can be represented as the infinite series

$$v(\vartheta,\varrho) = v_0(\vartheta,\varrho) + Hv_1(\vartheta,\varrho) + H^2 v_2(\vartheta,\varrho) + \cdots \quad , \tag{4.8}$$

$$U_k(\vartheta,\varrho) = U_{k_0}(\vartheta,\varrho) + HU_{k_1}(\vartheta,\varrho) + H^2 U_{k_2}(\vartheta,\varrho) + \cdots \quad . \tag{4.9}$$

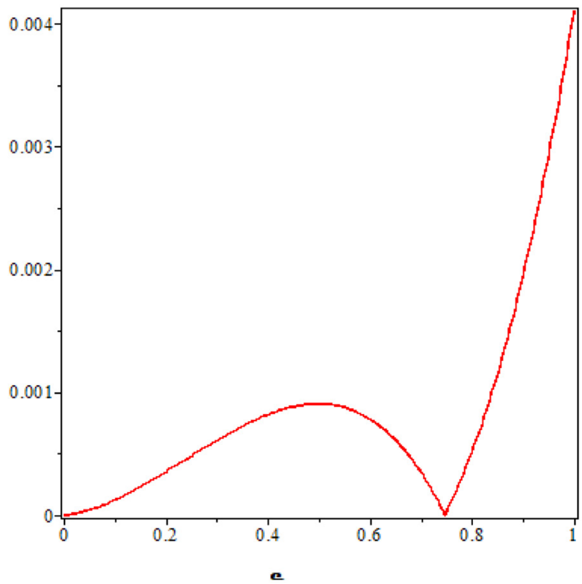
by substituting Eqs. (4.8) and (4.9) in Eqs. (4.6) and (4.7)respectively, and equating the terms of the same powers of $H$, we obtain the following set of linear equations:

$$H^0 : \begin{cases} v_0(\vartheta,\varrho) = \alpha(\vartheta,\varrho) \\ U_{k_0}(\vartheta,\varrho) = 0 \end{cases}$$

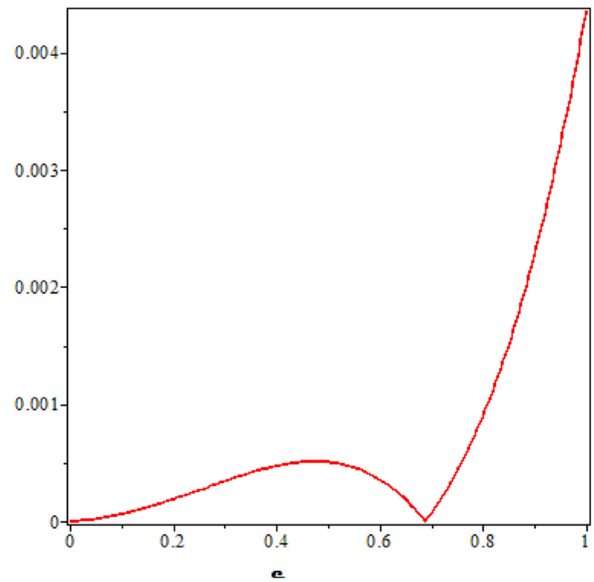
$$H^1 : \begin{cases} Lv_1 + Lv_0 + \sum_{k=1}^N B_k \varrho^{1-k-\delta(\vartheta,\varrho)} U_{k_0} - F(\vartheta,\varrho, v_0(p_0\vartheta, q_0\varrho), \frac{\partial v_0(p_1\vartheta, q_1\varrho)}{\partial\vartheta}, \ldots) = 0 \\ U_{k_1} + U_{k_0} - \int_0^t s^{k-1} \frac{\partial v_0}{\partial\varrho} ds = 0 \end{cases}$$

$$H^2 : \begin{cases} Lv_2 + \sum_{k=1}^N B_k \varrho^{1-k-\delta(\vartheta,\varrho)} U_{k_1} - F(\vartheta,\varrho, v_1(p_0\vartheta, q_0\varrho), \frac{\partial v_1(p_1\vartheta, q_1\varrho)}{\partial\vartheta}, \ldots) = 0 \\ U_{k_2} - \int_0^t s^{k-1} \frac{\partial v_1}{\partial\varrho} ds = 0 \end{cases}$$

$$H^3 : \begin{cases} Lv_3 + \sum_{k=1}^N B_k \varrho^{1-k-\delta(\vartheta,\varrho)} U_{k_2} - F(\vartheta,\varrho, v_2(p_0\vartheta, q_0\varrho), \frac{\partial v_2(p_1\vartheta, q_1\varrho)}{\partial\vartheta}, \ldots) = 0 \\ U_{k_3} - \int_0^t s^{k-1} \frac{\partial v_2}{\partial\varrho} ds = 0 \end{cases}$$




(a) $When\ \delta(\vartheta,\varrho) = 2 - 0.2sin(\vartheta\ \varrho)$




(b) $When\ \delta(\vartheta,\varrho) = 1.7 + 0.1e^{(-\vartheta\ \varrho)}$

**Fig. 1.** The absolute error of Example 6.1.

and so on. Now, by solving for the terms $v_i$ and $U_{k_i}, i = 1, 2, \ldots, k = 1, 2, \ldots$, Eqs. (4.8) and (4.9) can be determined. Meanwhile, by making the parameter $H$ goes to unity, one has

$$v(\vartheta,\varrho) = v_0 + v_1(\vartheta,\varrho) + v_2(\vartheta,\varrho) + \cdots, \tag{4.10}$$

$$U_k(\vartheta,\varrho) = U_{k_0}(\vartheta,\varrho) + U_{k_1}(\vartheta,\varrho) + U_{k_2}(\vartheta,\varrho) + \cdots, \tag{4.11}$$

which approximated the solution for problem (1.1) and (1.2)

## 5. Analysis of convergence and estimation of error

In this section, we focus on the convergence of the HPM for Eqs. (1.1) and (1.2). The sufficient conditions for convergence of the method and the error estimate are presented.[47,48]

**Theorem 5.1.** *Let $v_m(\vartheta,\varrho)$ and $v(\vartheta,\varrho)$ be defined in Banach space $(C[0,1], \|.\|)$. Then the series solution $\{v_m(\vartheta,\varrho)\}_{m=0}^{\infty}$ defined by Eq. (4.10) converges to the solution of Eq. (4.1).*

**Theorem 5.2.** *The maximum absolute truncation error of the series solution Eq. (4.10) for Eq. (4.1) is estimated to be*

$$\left| v(\vartheta,\varrho) - \sum_{i=0}^{m} v_i(\vartheta,\varrho) \right| \leq \frac{\beta^{m+1}}{(1-\beta)} \parallel v_0(\vartheta,\varrho) \parallel . \tag{5.1}$$

*where $0 < \beta < 1$.*

## 6. Illustrated examples

The HPM described in the preceding section will be utilized to some (VFPDDEs).

**Example 6.1.** Think about the solution of the following proportional delay generalized time-fractional Burgers equation:

$${}_0^C D_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) = \frac{\partial^2}{\partial\vartheta^2} v(\vartheta,\frac{\varrho}{3})\ v(\vartheta,\frac{\varrho}{3}) - \frac{1}{3} v(\vartheta,\varrho), \tag{6.1}$$

$\vartheta, \varrho \in [0,1]$ and $\delta(\vartheta,\varrho) = 2 - 0.3sin(\vartheta\ \varrho)$ for initial conditions $v(\vartheta,0) = \vartheta^2$. The exact solution of this equation is $v(\vartheta,\varrho) = \vartheta^2 cosh(1.3\varrho)$.

By employing the proposed technique given in Section 4, the first few components of the homotopy perturbation for Eq. (6.1) derived as follows:

$$\begin{aligned}
&v_0(\vartheta,\varrho) = \vartheta^2,\\
&v_1(\vartheta,\varrho) = a\ \vartheta^2 \varrho^{\delta(\vartheta,\varrho)},\\
&v_2(\vartheta,\varrho) = (b-c)\ \vartheta^2 \varrho^{2\delta(\vartheta,\varrho)},\\
&v_3(\vartheta,\varrho) = \left((b-c)d + 4ad3^{-2\delta(\vartheta,\varrho)}\right) \vartheta^2 \varrho^{3\delta(\vartheta,\varrho)},\\
&\quad\vdots
\end{aligned}$$

also,

$$\begin{aligned}
&U_{k_0}(\vartheta,\varrho) = U_{k_1}(\vartheta,\varrho) = 0,\\
&U_{k_2}(\vartheta,\varrho) = \frac{5}{3a}(2-\delta(\vartheta,\varrho))\,\vartheta^{2-\delta(\vartheta,\varrho)},\\
&U_{k_3}(\vartheta,\varrho) = \frac{25}{9a^2}\left(4 - \delta(\vartheta,\varrho)\varrho^{4-\delta(\vartheta,\varrho)} - \vartheta^2\varrho^2\right),\\
&\quad\vdots
\end{aligned}$$

The $n$th-order approximate solution of Eq. (6.1)

$$v(\vartheta,\varrho) \simeq v_0(\vartheta,\varrho) + v_1(\vartheta,\varrho) + v_2(\vartheta,\varrho) + v_3(\vartheta,\varrho) + \cdots + v_n(\vartheta,\varrho). \tag{6.2}$$

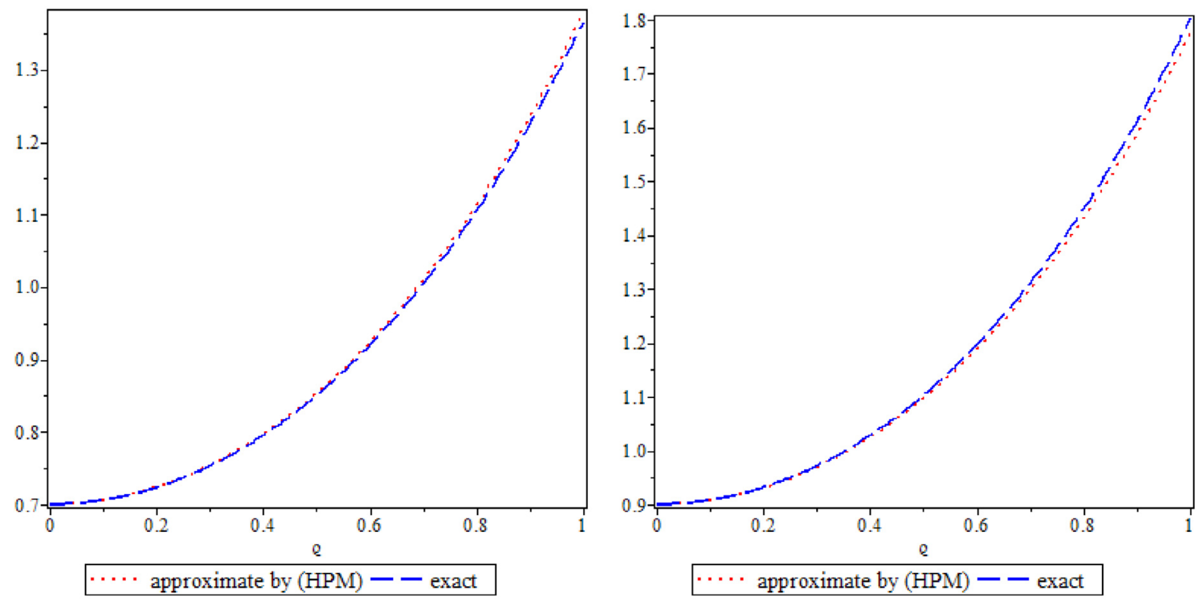
Where

$$a = \frac{5}{3\ \Gamma(\delta(\vartheta,\varrho)+1)}, \quad b = \frac{20 \times 3^{-1-\delta(\vartheta,\varrho)}}{\Gamma(2\delta(\vartheta,\varrho)+1)},$$
$$c = \frac{5}{3\ \Gamma(\delta(\vartheta,\varrho)+1)}, \quad d = \frac{\Gamma(2\ \delta(\vartheta,\varrho)+1)}{\Gamma(3\ \delta(\vartheta,\varrho)+1)},$$

Figs. 1 and 2 represent the absolute error and the approximate solution of Example 6.1 for $N = 3$ and different values of $\delta(\vartheta,\varrho)$ at $\vartheta = 0.9$ respectively. Finally the absolute error for different values of $(\vartheta,\varrho)$ is given in Table 1.

**Example 6.2.** Consider the following proportional delay generalized time-fractional Burgers equation:

$${}_0^C D_\varrho^{\delta(\vartheta,\varrho)} v(\vartheta,\varrho) = \frac{\partial^2}{\partial\vartheta^2} v(\vartheta,\varrho) + v(\vartheta,\frac{\varrho}{2}) v(\frac{\vartheta}{2},\frac{\varrho}{2}) + \frac{1}{2} v(\vartheta,\varrho), \tag{6.3}$$




(a) When $\delta(\vartheta,\varrho)=2-0.3sin(\vartheta\ \varrho)$  (b) When $\delta(\vartheta,\varrho)=1.7+0.1e^{(-\vartheta\ \varrho)}$

**Fig. 2.** Numerical results of Example 6.1 at $\vartheta=0.9$.

**Table 1**
Absolute error for Example 6.1 by $\delta(\vartheta,\varrho)=2-0.2sin(\vartheta\ \varrho)$.

| $(\vartheta,\varrho)$ | Absolute error |
|---|---|
| (0.25,0.25) | $5.865\times10^{-4}$ |
| (0.25,0.50) | $1.775\times10^{-4}$ |
| (0.25,0.75) | $1.903\times10^{-4}$ |
| (0.50,0.25) | $8.265\times10^{-3}$ |
| (0.50,0.50) | $3.181\times10^{-4}$ |
| (0.50,0.75) | $9.717\times10^{-4}$ |
| (0.75,0.25) | $3.233\times10^{-4}$ |
| (0.75,0.50) | $1.458\times10^{-3}$ |
| (0.75,0.75) | $1.903\times10^{-3}$ |

**Table 2**
Absolute error for Example 6.2 by $\delta(\vartheta,\varrho)=1.7+0.3cos^2(\vartheta\ \varrho)$.

| $(\vartheta,\varrho)$ | Absolute error |
|---|---|
| (0.25,0.25) | $8.682\times10^{-4}$ |
| (0.25,0.50) | $1.549\times10^{-4}$ |
| (0.25,0.75) | $8.689\times10^{-3}$ |
| (0.50,0.25) | $4.608\times10^{-3}$ |
| (0.50,0.50) | $1.654\times10^{-4}$ |
| (0.50,0.75) | $1.179\times10^{-3}$ |
| (0.75,0.25) | $2.496\times10^{-4}$ |
| (0.75,0.50) | $1.072\times10^{-4}$ |
| (0.75,0.75) | $4.482\times10^{-4}$ |

with initial conditions $v(\vartheta,0)=\vartheta,\ \delta(\vartheta,\varrho)=1.7+0.3cos^2(\vartheta\ \varrho)$. The exact solution of this problem is $v(\vartheta,\varrho)=\vartheta e^{0.5\varrho}$.

Similarly using the approach given in Section 4, the first components of the homotopy perturbation for Eq. (6.3) are derived as follows:

$$v_0(\vartheta,\varrho)=\vartheta,$$
$$v_1(\vartheta,\varrho)=a\ \vartheta\varrho^{\delta(\vartheta,\varrho)},$$
$$v_2(\vartheta,\varrho)=a\ b\ \vartheta\ \varrho^{2\delta(\vartheta,\varrho)},$$
$$v_3(\vartheta,\varrho)=(a\ b\ c+a\ d)\,\vartheta\ \varrho^{3\delta(\vartheta,\varrho)},$$
$$\vdots$$

also,

$$U_{k_0}=U_{k_1}=0,$$
$$U_{k_2}=2\ a\ b\vartheta\varrho\ e^{\delta(\vartheta,\varrho)}\ cos(2\vartheta\varrho),$$
$$U_{k_3}=a\ \vartheta^2\varrho\ e^{\delta(\vartheta,\varrho)}cos(2\vartheta\varrho)+2\ a\ b\vartheta\varrho^3\ e^{\delta(\vartheta,\varrho)}\ cos(2\vartheta\varrho),$$
$$\vdots$$

where

$$a=\frac{1}{\Gamma(\delta(\vartheta,\varrho)+1)},\quad b=\frac{2^{\delta(\vartheta,\varrho)}+2^{-1}}{\Gamma(2\delta(\vartheta,\varrho)+1)},$$

$$c=\frac{2^{-2\delta(\vartheta,\varrho)}}{\Gamma(3\delta(\vartheta,\varrho)+1)},\quad d=\frac{2^{-1-2\delta(\vartheta,\varrho)}}{\Gamma(3\delta(\vartheta,\varrho)+1)},$$

The rest parts of the homotopy perturbation solution can be produced in the same way for the subsequent components. The following is the approximate $n$th-order solution of Eq. (6.3):

$$v(\vartheta,\varrho)\simeq v_0(\vartheta,\varrho)+v_1(\vartheta,\varrho)+v_2(\vartheta,\varrho)+v_3(\vartheta,\varrho)+\cdots+v_n(\vartheta,\varrho). \quad (6.4)$$

Figs. 3 and 4 represent the approximate solution and the absolute error of Example 6.2 for $N=3$ and different values of $\delta(\vartheta,\varrho)$ at $\vartheta=0.7$ respectively. Finally the absolute error for different values of $(\vartheta,\varrho)$ is given in Table 2.

**Example 6.3.** Given the following FPDE with proportional delay:

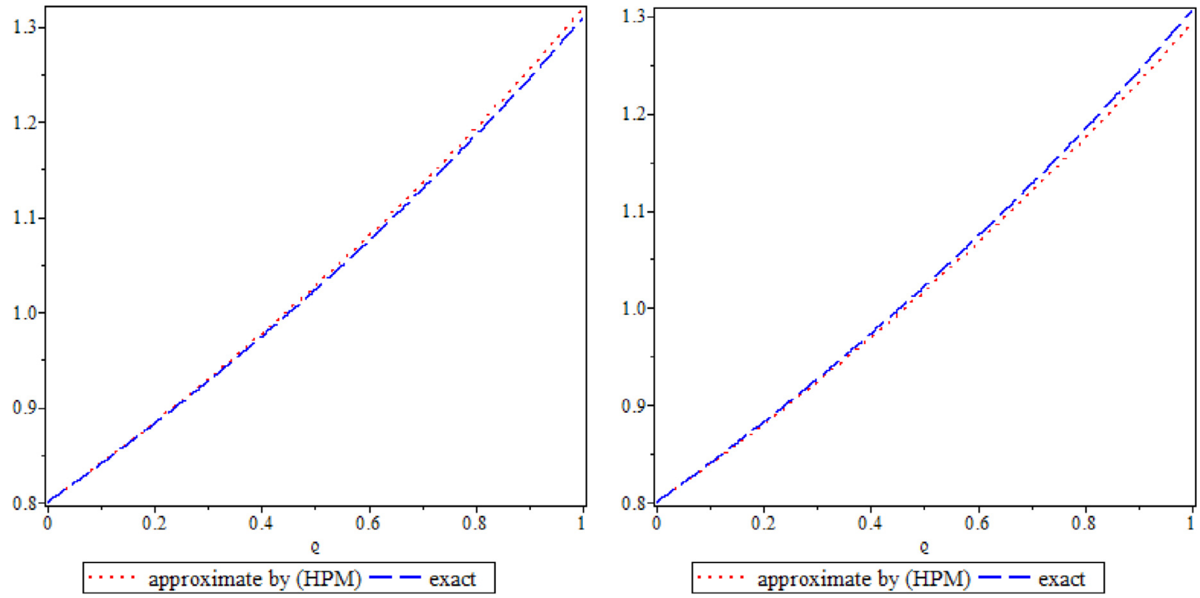
$${}_0^CD_\varrho^{\delta(\vartheta,\varrho)}v(\vartheta,\varrho)=\frac{\partial^2}{\partial\vartheta^2}v(\frac{\vartheta}{2},\frac{\varrho}{2})\frac{\partial}{\partial\vartheta}v(\frac{\vartheta}{2},\frac{\varrho}{2})-\frac{1}{8}\frac{\partial}{\partial\vartheta}v(\vartheta,\varrho)-v(\vartheta,\varrho), \quad (6.5)$$

$\vartheta,\varrho\in[0,1]$ and $\delta(\vartheta,\varrho)=2-0.3\ exp(\vartheta\ \varrho)$ with elementary conditions $v(\vartheta,0)=\vartheta^2$.

The exact solution of this problem is $v(\vartheta,\varrho)=\vartheta^2cos(\varrho)$.

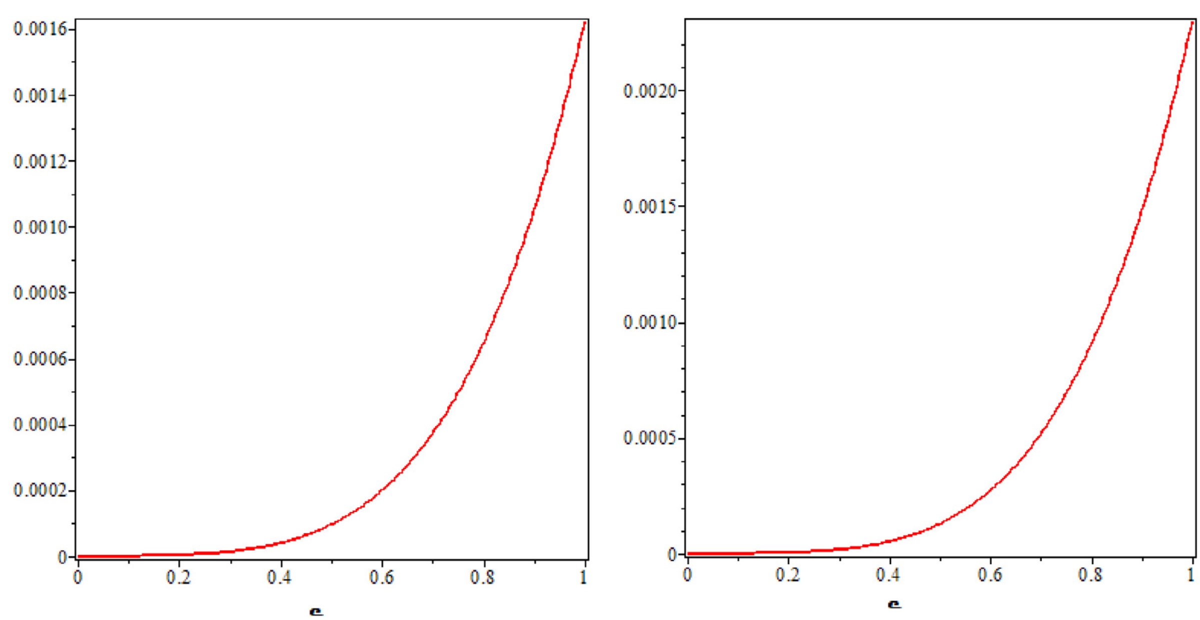
The residuum parts of the homotopy perturbation solution can be produced in the same way for the subsequent components. The approximate $n$th-order solution to Eq. (6.5).

$$v(\vartheta,\varrho)\simeq v_0(\vartheta,\varrho)+v_1(\vartheta,\varrho)+v_2(\vartheta,\varrho)+v_3(\vartheta,\varrho)+\cdots+v_n(\vartheta,\varrho). \quad (6.6)$$



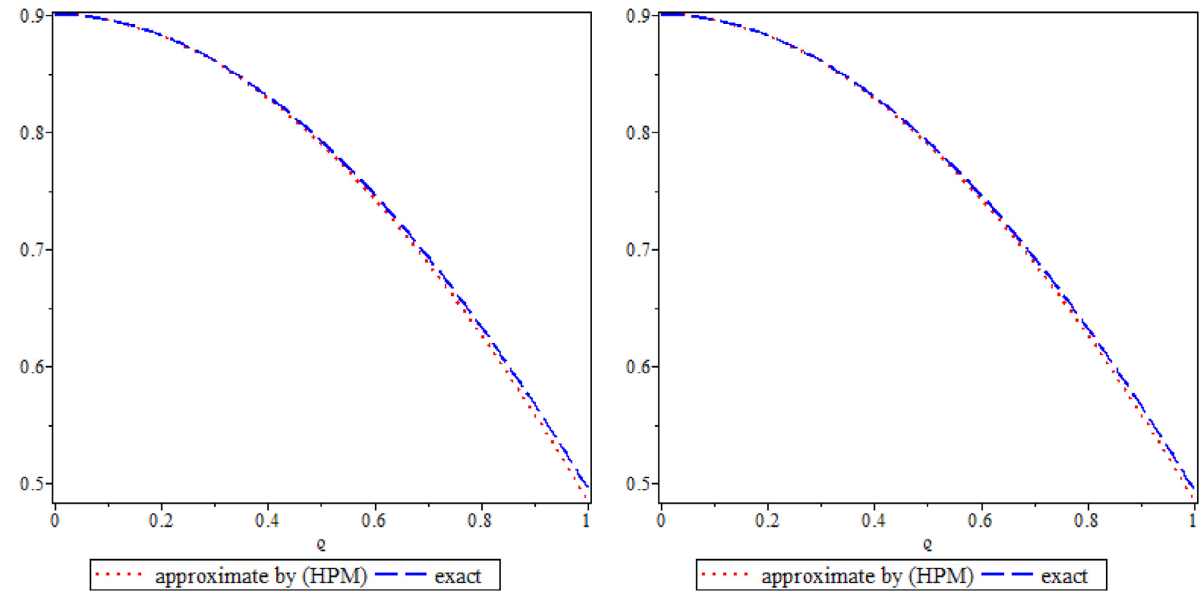

(a) $When\ \delta(\vartheta, \varrho) = 1.7 + 0.3cos^2(\vartheta\ \varrho)$ (b) $When\ \delta(\vartheta, \varrho) = 1.7 + 0.3cos^2(\vartheta\ \varrho)$

**Fig. 3.** Numerical results of Example 6.2.



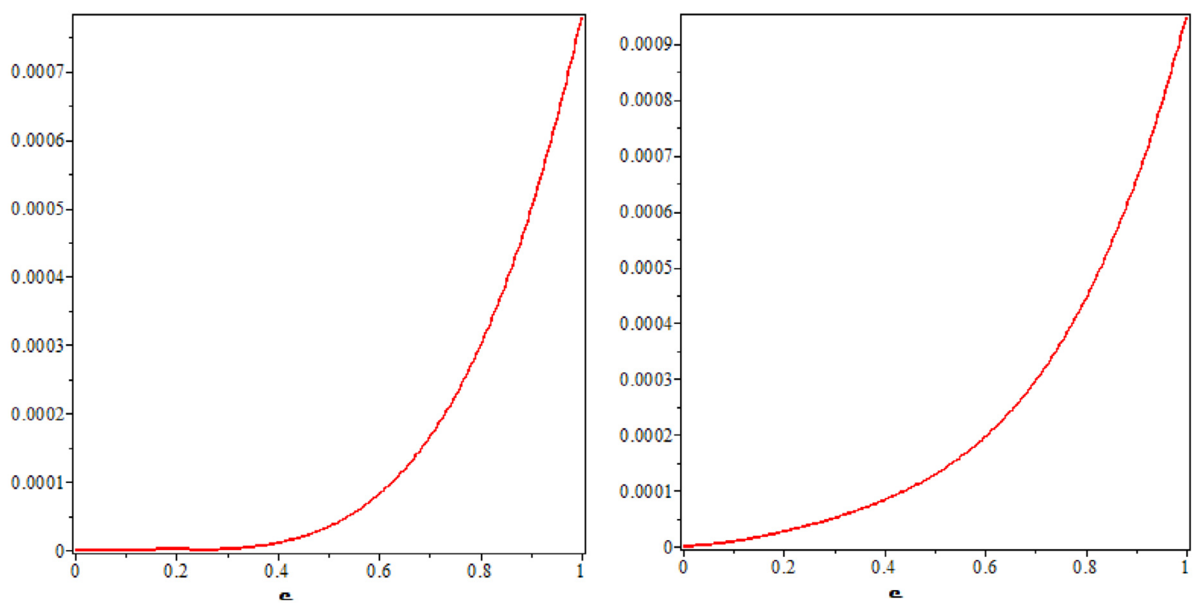
(a) $When\ \delta(\vartheta, \varrho) = 1.7 + 0.3cos^2(\vartheta\ \varrho)$ (b) $When\ \delta(\vartheta, \varrho) = 2 + 0.2sin(-\vartheta\ \varrho)$

**Fig. 4.** The absolute error of Example 6.2 at $\vartheta = 0.7$.




(a) $When\ \delta(\vartheta,\varrho) = 2 - 0.2sin(\vartheta\ \varrho)$ (b) $When\ \delta(\vartheta,\varrho) = 1.7 + 0.3e^{(-\vartheta\ \varrho)}$

**Fig. 5.** Numerical results of Example 6.3.




(a) $When\ \delta(\vartheta,\varrho) = 2 - 0.2sin(\vartheta\ \varrho)$ (b) $When\ \delta(\vartheta,\varrho) = 1.7 + 0.3e^{(-\vartheta\ \varrho)}$

**Fig. 6.** The absolute error of Example 6.3 at $\vartheta = 0.8$.

**Table 3**
Absolute error for Example 6.3 by $\delta(\vartheta,\varrho) = 2 - 0.3e^{(\vartheta\ \varrho)}$.

| $(\vartheta,\varrho)$ | Absolute error |
|---|---|
| (0.25,0.25) | $1.817 \times 10^{-4}$ |
| (0.25,0.50) | $5.291 \times 10^{-4}$ |
| (0.25,0.75) | $1.649 \times 10^{-4}$ |
| (0.5,0.25) | $3.871 \times 10^{-3}$ |
| (0.50,0.50) | $3.986 \times 10^{-4}$ |
| (0.50,0.75) | $2.949 \times 10^{-4}$ |
| (0.75,0.25) | $1.220 \times 10^{-4}$ |
| (0.75,0.50) | $2.670 \times 10^{-3}$ |
| (0.75,0.75) | $4.087 \times 10^{-4}$ |

Figs. 5 and 6 represent the approximate solution and the absolute error of Example 6.3 for $N = 3$ and different values of $\delta(\vartheta,\varrho)$ at $\vartheta = 0.7$ respectively. Finally the absolute error for different values of $(\vartheta,\varrho)$ is given in Table 3.

## 7. Conclusions

The HPM is used in this study to approximate the solutions of VF-PDEs with proportional delays. The variable order fractional derivative was approximated in terms of the standard derivative. Comparatively speaking, the proposed method requires a lot less computational work than other numerical methods. The obtained results, compared with exact solution, show us that this method is remarkable very effective, very simple, and very fast in convergence for handling VFPDEs with proportional delays. In this paper Maple 2016 was used for all calculations.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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