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# An efficient numerical algorithm for solving delay differential equations based on second kind Chebyshev polynomials

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## ABSTRACT

This paper proposes a fast spectral technique based on second kind Chebyshev polynomials (SKCPs) to numerically solve delay differential equations (DDEs). First, we introduce some features of the SKCPs. Then, we use the operational matrices of coefficients, derivation, and stretch of the shifted SKCPs to convert DDEs into systems of algebraic equations. We show that these matrices are sparse, allowing for a fast implementation of the numerical calculations. A theoretical discussion about error estimation is conducted, and finally, numerical examples are given to highlight the accuracy and efficiency of the proposed method.



## I. INTRODUCTION

Delay differential equations (DDEs) are relevant to develop realistic mathematical models since many processes depend on past history.[1–4] Indeed, DDEs play a significant role in modeling and analyzing many problems arising in physiological processes, population dynamics, chemical kinetics, and the evolution of infections,[5–9] to cite a few. Delay models were introduced in engineering by Minorsky for ship stabilization and by Dietrich and von Schlippe for wheel shimmy modeling.[10,11] Theorems and results for DDEs are quite well-developed in the literature.[12–14] Time delay is also a key element in machine tool chatter.[15] Many advanced models have recently emerged for milling and turning applications.[16] Therefore, applications and performance of DDEs in various branches of engineering and science, such as communication networks, transmission lines, population dynamics, and biological systems, have interested researchers, leading to the development of approximation methods,[17–20] whether analytical or numerical. Heffernan and Corless[21] solved DDEs by computer algebra, while Behroozifar and Yousefi[22] proposed the operational matrices of Bernstein polynomials and hybrid block-pulse function. Cakir and Arslan[23] implemented the differential transform and Adomian decomposition methods to approximate multi-pantograph DDEs. Hwang and Chen[24] used shifted Legendre polynomials for time-delay systems. Hafshejani *et al.*[25] adopted the Legendre wavelet approximation technique.

Many DDEs are difficult or impossible to solve analytically, and thus, numerical approaches are required. Since numerical solutions are always approximations to the real solutions, minimizing the error is essential. Current numerical methods present limitations, and the development of faster and more accurate approaches is a matter of great interest.[26,27] In this paper, we introduce a fast spectral method based on second kind Chebyshev polynomials (SKCPs) for the numerical solution of DDEs, given by

$$\begin{cases} \dfrac{d}{dt}y(t) = q(t)y(t) + p(t)y\left(\dfrac{t}{\lambda}\right), & t \in [0, t_f], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

in which $p(t)$ and $q(t)$ represent prescribed continuous functions over $I(t_f) := [0, t_f]$, $\lambda > 1$ is a stretched argument, and $y_0$ represents the initial condition (IC). These equations are very important in modeling some real-world problems, such as the $CO_2$ gas cycle in the human body.

Initially, we outline the key properties of the SKCPs. Subsequently, we employ the operational matrices for the coefficients, derivatives, and stretching of the shifted SKCPs to transform the DDEs into systems of algebraic equations. The sparsity of these matrices is demonstrated, enabling a swift implementation of the numerical computations. We also provide a theoretical analysis of error estimation and conclude with numerical examples that illustrate the accuracy and efficiency of the proposed approach.

The layout of this paper is detailed as follows: Section II introduces the notation and basic definitions of SKCPs. Section III presents the operational matrices, including derivative, stretch, and product of SKCPs. Section IV proposes the computational technique to approximate DDEs (1.1). Section V presents the error analysis of the proposed scheme. Section VI discusses four examples to illustrate the accuracy and applicability of the method. Finally, Sec. VII summarizes the main findings.

## II. THE SKCPS AND THEIR PROPERTIES

### A. Shifted Chebyshev polynomials of the second kind

This subsection presents important definitions and properties of SKCPs, which will be used in solving DDEs (1.1) with the IC $y(0) = y_0$.[28–30] A set of delay differential equations is defined in the range $I(T) := [0, T]$. For approximating and analyzing these equations by using SKCPs originally defined over the interval $[-1, 1]$, we use the following linear transformation:

$$x = \frac{b-a}{2}t + \frac{b+a}{2}, \quad x \in [a, b], \quad t \in [-1, 1]. \tag{2.1}$$

Substituting $a = 0$ and $b = T$ in Eq. (2.1), we obtain

$$x = \frac{T}{2}t + \frac{T}{2} \quad \text{or} \quad t = \frac{2}{T}x - 1. \tag{2.2}$$

*Definition 1.* A set $\{\varphi_i(t) : i = 0, 1, \ldots\}$ of shifted SKCPs can be defined on $I(T) := [0, T]$ by[31]

$$\varphi_i(t) = U_i\left(\frac{2}{T}t - 1\right), \quad i = 0, 1, 2, \ldots, \tag{2.3}$$

where $U_i(x)$ stands for the SKCP with order $i$ and meets the recurrence condition,

$$U_i(t) = 2tU_{i-1}(t) - U_{i-2}(t), \quad U_0(t) = 1,$$
$$U_1(t) = 2t, \quad i = 2, 3, \ldots.$$

The SKCPs $U_i(x)$ are orthogonal with respect to $L^2$, with the inner product over the domain $[-1, 1]$ and weight function given by $\overline{w}(t) = \sqrt{1 - t^2}$. The SKCPs are widely used due to their properties for the approximation of the function. The analytical form of the shifted SKCPs of degree $i$ is as follows:[31]

$$\varphi_i(t) = \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^k}{(i-k)!(2k+1)!T^k} t^k. \tag{2.4}$$

Some properties of shifted SKCPs are

$$\varphi_i(t)\varphi_j(t) = \sum_{k=0}^{\frac{i+j-|i-j|}{2}} \varphi_{i+j-2k}(t),$$
$$\times \int_0^T w(t)\varphi_i(t)\varphi_j(t)dt = \frac{T\pi}{4}\delta_{ij},$$

with $w(t) = \sqrt{1 - \left(\frac{2}{T}t - 1\right)^2}$ representing the weight function.

### B. Function approximation

Any function $y(t) \in L^2[0, 1]$ can be expanded using shifted SKCPs as[31]

$$y(t) = \sum_{i=0}^{\infty} a_i\varphi_i(t). \tag{2.5}$$

Therefore, by truncating the infinite series in Eq. (2.5), we can approximate $y(t)$ as

$$y(t) \simeq \sum_{i=0}^{N} a_i\varphi_i(t) = \varphi(t)^T A = A^T\varphi(t), \tag{2.6}$$

where

$$A = [a_0, a_1, a_2, \ldots, a_N]^T,$$
$$\varphi(t) = [\varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T,$$

with the coefficients $a_i$ for $i = 0, 1, 2, \ldots, N$ being given by

$$a_i = \frac{4}{T\pi}\int_0^T w(t)y(t)\varphi_i(t)dt. \tag{2.7}$$

**Theorem 1.** Suppose that $g \in H^k[-1, 1]$ (Sobolev space) and $\sum_{i=0}^{M} g_i\varphi_i(t)$ is

$$\chi_{ij} = \langle\varphi_{i-1}(x), \zeta(x,t)\varphi_{j-1}(t)\rangle$$
$$= \frac{16}{\pi^2T^2}\int_{-1}^{1}\int_0^1 w(x)w(t)\varphi_{i-1}(x)\varphi_{j-1}(t)\zeta(x,t)dtdx, \tag{2.8}$$

$$\chi_{ij} \simeq \sum_{k=0}^{N}\sum_{l=0}^{N} w_{1l}w_{2k}\varphi_{i-1}(x_l)\varphi_{j-1}(x_k)\zeta(x_l, x_k). \tag{2.9}$$

In Eq. (2.9), the symbols $x_k$ and $x_l$ represent the roots of the Legendre polynomial $P_{N+1}(t)$, and the weights $w_{1l}$ and $w_{2k}$ are obtained as follows:

$$w_l = w_{2l} = \frac{2}{(1-x_l^2)[P'_{N+1}(x_l)]^2}, \quad 0 \le l \le N. \tag{2.10}$$

**Theorem 2 (See Ref. 32).** Suppose that $g(t) \in H^k([-1,1])$ (Sobolev space) and let $\sum_{i=0}^{N} g_i\varphi_i(t)$ be the best approximation polynomial of $g$ in $L_2$-norm. Therefore, we have

$$\left\| g(t) - \sum_{i=0}^{N} g_i\varphi_i(t) \right\|_{L_2[-1,1]} \le C_0 N^{-k} \|g(t)\|_{H^k([-1,1])}, \tag{2.11}$$

in which $C_0 \in \mathbb{R}^+$ is independent of $g(t)$ and $N$ and is dependent on the selected norm.

## III. OPERATIONAL MATRICES OF SKCPS

This section introduces the stretch and first derivative operational matrices that are used to construct the proposed method. Given that

$$\frac{d}{dt}\varphi(t) = D^{(1)}\varphi(t), \tag{3.1}$$

we approximate

$$\frac{d}{dt}\varphi_i(t), \quad i = 0,1,\ldots,N \quad \text{by} \quad \sum_{j=0}^{N} a_j^{(1)}\varphi_j(t), \tag{3.2}$$

where

$$\begin{aligned} a_{ij}^{(1)} &= \left\langle \frac{d}{dt}\varphi_i(t), \varphi_j(t) \right\rangle \\ &= \frac{4}{T\pi}\int_0^T w(t)\left(\frac{d}{dt}\varphi_i(t)\right)\varphi_j(t)dt, \quad i,j = 1,\ldots,N, \end{aligned} \tag{3.3}$$

and $D^{(1)}$ is the $(N+1)\times(N+1)$ square operational matrix of the first derivative for our basis $\varphi(t)$, defined by

$$D^{(1)} = (d_{ij}) = \begin{cases} 4j, & k = 1,2,\ldots,N \ \text{ if } N \text{ even}, \quad k = 1,2,\ldots,N-1 \ \text{ if } N \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}$$

with $i,j = 0,1,\ldots,N$. For example, for $N = 4, 5$, we have

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 4 & 0 & 12 & 0 & 0 \\ 0 & 8 & 0 & 16 & 0 \end{bmatrix}_{5\times5}, \quad D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 4 & 0 & 12 & 0 & 0 & 0 \\ 0 & 8 & 0 & 16 & 0 & 0 \\ 4 & 0 & 12 & 0 & 20 & 0 \end{bmatrix}_{6\times6}.$$

By applying Eq. (3.1), it is clear that

$$\frac{d^n}{dt^n}\varphi(t) = (D^{(1)})^n\varphi(t) = D^{(n)}\varphi(t), \quad n = 0,1,2,\ldots,$$

where the superscript in $D^{(1)}$ denotes matrix powers and $n \in \mathbb{N}$.

The operational matrix of SKCPs can be introduced as a matrix $S$ that satisfies the equation

$$\varphi\left(\frac{t}{\lambda}\right) = S\varphi(t). \tag{3.4}$$

It is obvious that

$$\varphi\left(\frac{t}{\lambda}\right) = \left[\varphi_0\left(\frac{t}{\lambda}\right), \varphi_1\left(\frac{t}{\lambda}\right), \varphi_2\left(\frac{t}{\lambda}\right), \ldots, \varphi_N\left(\frac{t}{\lambda}\right)\right]^T. \tag{3.5}$$

We approximate

$$\varphi_n\left(\frac{t}{\lambda}\right) \tag{3.6}$$

by

$$\sum_{i=1}^{N} a_{ni}\varphi_i(t) = A_n^T\varphi_p(t), \quad n,i = 0,1,\ldots,N, \tag{3.7}$$

where

$$\begin{aligned} a_{ni} &= \left\langle \varphi_n\left(\frac{t}{\lambda}\right), \varphi_i(t) \right\rangle \\ &= \frac{4}{T\pi}\int_0^T w(t)\varphi_n\left(\frac{t}{\lambda}\right)\varphi_i(t)dt, \quad n,i = 1,2,\ldots,N. \end{aligned} \tag{3.8}$$

Hence,

$$S = \left[A_0^T, A_1^T, A_2^T, \ldots, A_N^T\right]. \tag{3.9}$$

For $\lambda = 2$ and $N = 5$, we have

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.25 & -1 & 0.25 & 0 & 0 & 0 \\ 0.25 & 0.75 & -0.75 & 0.125 & 0 & 0 \\ -0.125 & 0 & 0.9375 & -0.5 & 0.0625 & 0 \\ -0.125 & -0.343\,75 & -0.4375 & 0.875 & -0.3125 & 0.031\,25 \end{bmatrix}.$$

For the numerical solution of differential and integral equations, we need to evaluate $\varphi(t)\varphi^T(t)$ in terms of $\varphi_i(t), i = 0,1,\ldots,N$. Let us assume that $C$ is an arbitrary $N \times 1$ vector and that

$$\varphi(t)\varphi^T(t)C \simeq \widetilde{C}\varphi(t), \tag{3.10}$$

where $\widetilde{C}$ is called the operational matrix of product. We have

$$\varphi(t)\varphi^T(t)C = \begin{bmatrix} \varphi_0(t)\varphi^T(t)C \\ \varphi_1(t)\varphi^T(t)C \\ \vdots \\ \varphi_N(t)\varphi^T(t)C \end{bmatrix} \simeq \widetilde{C}\varphi(t), \tag{3.11}$$

where the elements of the product operational matrix $\widetilde{C} = [\widetilde{c}_{ij}]$, $i,j = 0,1,2,\ldots,N$, of dimension $(N+1)\times(N+1)$ are[31]

$$\widetilde{c}_{ij} = \sum_{k=\max\left(0,\frac{i+j-N}{2}\right)}^{\frac{i+j-|i-j|}{2}} c_{i+j-2k}. \tag{3.12}$$

## IV. THE SPECTRAL METHOD FOR SOLVING DDEs

This section adopts the numerical scheme to approximate the DDEs (1.1) with the IC given by $\mathcal{Y}(0) = y_0$. From (2.6), we approximate $\mathcal{Y}$ by

$$y(t) \simeq A^T \varphi(t). \tag{4.1}$$

Considering Eq. (3.1), we obtain

$$\frac{d}{dt} y(t) \simeq A^T D\varphi(t). \tag{4.2}$$

In addition, we use

$$\varphi\left(\frac{t}{\lambda}\right) \simeq S\varphi(t), \tag{4.3}$$

in which $D$ and $S$ are introduced in Eqs. (3.1) and (3.4). By using Eqs. (4.1)–(4.3) into the main problem (1.1), we obtain

$$A^T DP(t) = P(t)S\varphi(t) + q(t)A^T\varphi(t). \tag{4.4}$$

In addition, we need to use the IC in Eq. (1.1), that is,

$$A^T\varphi(0) = y_0. \tag{4.5}$$

Finally, Eqs. (4.4) and (4.5) give a system of linear equations. We collocate Eq. (4.5) in the set of $N - 1$ nodal points $t_l$ of the Gauss–Chebyshev form[33] as

$$Q_n = \{t_l | P_{n+1,0}(t_l) = 0, \quad l = 0, 1, \ldots, n\}. \tag{4.6}$$

By substituting the nodes $t_l$ in (4.4), we have

$$A^T\varphi(t_l) = P(t_l)S\varphi(t_l) + q(t_l)A^T\varphi(t_l), \quad l = 0, 1, \ldots, N - 1. \tag{4.7}$$

We can solve a system of $N$ linear equations by combining Eqs. (4.5) and (4.7). Then, we use the vector $A$ for computing the numerical solution as

$$y(t) \simeq A^T\varphi(t), \quad 0 \le t \le 1. \tag{4.8}$$

## V. ERROR ESTIMATION

Herein, we estimate the approximation error of the solution and obtain an upper error norm bound for the numerical technique. For this aim, we consider the DDE as

$$\begin{cases} y'(t) = ay(t) + by(rt), \\ y(0) = y_0, \quad t \in [0, 1], \end{cases} \tag{5.1}$$

where $y \in C^1[a, b]$ and $0 < r < 1$. We also consider an arbitrary function $f \in L^2[0, 1]$ and use Taylor polynomials of order $M - 1$, defined on $[a, b]$. To approximate the arbitrary function $f(t)$, for computing the truncation error, we use the Taylor's residual theorem. Therefore, we have

$$e_n(t) = f(t) - \sum_{i=0}^{M-1} \frac{f^{(i)}(a)}{i!}(t - a)^i = \frac{(t - a)^M}{M!} f^{(M)}(\xi), \tag{5.2}$$

where $a < \xi < t$ and

$$\|e_r\|_\infty \le \frac{(b - a)^M}{M!} \left\| f^{(M)} \right\|_\infty. \tag{5.3}$$

By considering the SKCPs on the domain [0, 1], we have the truncation as[34]

$$\|e_n(t)\|_\infty \le \frac{1}{M!} \left\| f^{(M)}(t) \right\|_\infty. \tag{5.4}$$

By applying $\int_0^t$ over both sides of the DDE (5.1), it results

$$\int_0^t y'(s)ds = \int_0^t ay(s)ds + \int_0^t by(rs)ds, \quad 0 \le t \le 1. \tag{5.5}$$

Therefore, we have

$$y(t) - y(0) = a\int_0^t y(s)ds + b\int_0^t y(rs)ds, \tag{5.6}$$

or

$$0 = y(t) - y(0) - a\int_0^t y(s)ds - b\int_0^t y(rs)ds. \tag{5.7}$$

On the other hand, by using (2.11), we obtain

$$\left\| y - \sum_{i=0}^N a_i\varphi_i(t) \right\|_\infty \le \frac{1}{M!} \left\| f^{(M)}(t) \right\|_\infty. \tag{5.8}$$

Therefore, if we show the function approximation error with the form

$$e_j = \left| y(t) - \sum_{i=0}^N a_i\varphi_i(t) \right|,$$

then we can rewrite (5.8) as

$$e_J \le \frac{1}{M!} \left\| f^{(M)}(t) \right\|_\infty. \tag{5.9}$$

On the other hand, we have

$$y(t) - y(0) \simeq y_J(t) - y_J(0). \tag{5.10}$$

Therefore, using (5.4), we get

$$e(t) = y_J(t) - y_0 - a\int_0^t y_J(s)ds - b\int_0^t y_J(rs)ds, \quad 0 \le t \le 1, \tag{5.11}$$

where $y_J(t)$ represents the approximate function of $y(t)$. Hence, in view of Eqs. (2.6) and (5.11), we get

$$e(t) = A^T\varphi(t) - y(0) - a\int_0^t A^T\varphi(s)ds - b\int_0^t A^T S\varphi(s)ds, \quad 0 \le t \le 1. \tag{5.12}$$

Now, we reduce the relationship (5.12) from Eq. (5.7) as

$$e(t) = 0 = A^T\varphi(t) - y(t) - a\left(\int_0^t (A^T\varphi(s) - y(s))ds\right) - b\left(\int_0^t (A^T S\varphi(s) - y(rs))ds\right). \tag{5.13}$$

Thus, we have

$$e(t) = e_J(t) - a\left(\int_0^t e_J(s)ds\right) - b\left(\int_0^t e_J(rs)ds\right). \tag{5.14}$$

Based on the absolute function property, we obtain from (5.14)

$$|e_J(t)| \le |e_J(t)| + a\left|\int_0^t e_J(s)ds\right| + b\left|\int_0^t e_J(rs)ds\right|. \tag{5.15}$$

In addition, we have

$$|e_J(t)| \le \max_{t\in[0,1]} |e_J(t)|. \tag{5.16}$$

By applying (5.16), we obtain

$$|e(t)| \le (1 + a + b)\max_{t\in[0,1]} |e_J(t)| = \gamma \max_{t\in[0,1]} |e_J(t)|, \tag{5.17}$$

where $\gamma = 1 + a + b$. So, we can write

$$\gamma \max_{t\in[0,1]} |e_J(t)| \le \gamma \frac{1}{M!}\left\|f^{(M)}\right\|_\infty. \tag{5.18}$$

Finally, we have

$$|e(t)| \le \frac{1}{M!}\left\|f^{(M)}\right\|_\infty, \quad 0 \le t \le 1. \tag{5.19}$$

## VI. ILLUSTRATIVE EXAMPLES

This section introduces four numerical examples to illustrate the accuracy, applicability, and effectiveness of the proposed method. For this aim, we introduce the following error norms:

$$e_\infty = \|y - y_n\|_\infty = \max_{0\le s\le 1} |y - y_n|, \tag{6.1}$$

$$e_n = \max_{0\le s\le 1} |y(t) - y_n(t)|, \tag{6.2}$$

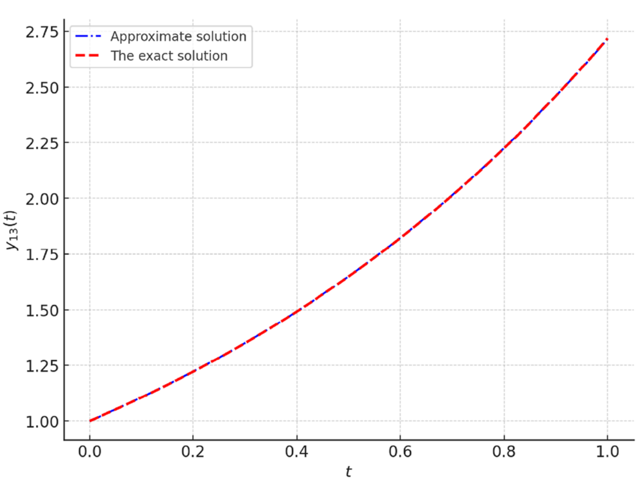


FIG. 1. Numerical and exact solutions by taking $N = 12$ and $\lambda = 2$ for example 1.

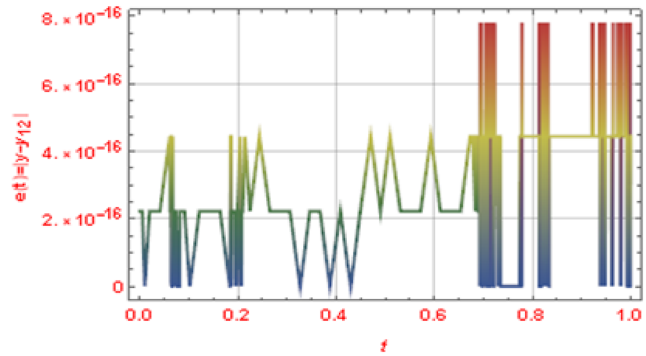


FIG. 2. Plot of the maximum norm errors by choosing $N = 12$ and $\lambda = 2$ for example 1.

$$\epsilon_n = \log_2\left(\frac{e_n}{e_{2n}}\right), \tag{6.3}$$

$$\zeta_n = \left(\int_0^1 w(t)e_n(t)dt\right)^{1/2}, \tag{6.4}$$

where $y$ and $y_n$ represent the exact and approximate solutions, respectively. Furthermore, all numerical results are calculated by the software Mathematica 11 on a personal computer with 4 GB RAM.

*Example 1.* Let us consider the following DEE:[25,35,36]

$$\begin{cases} \dfrac{d}{dt}y(t) = \dfrac{1}{2}e^{t/2}y\left(\dfrac{t}{2}\right) + \dfrac{1}{2}y(t), & t \in [0,1],\\ y(0) = 1, \end{cases} \tag{6.5}$$

where the analytic solution is $y = \exp(t)$.

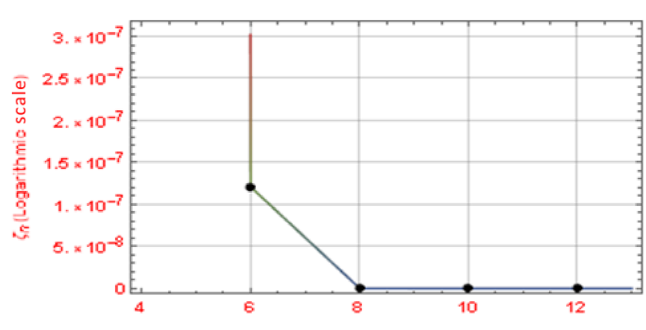


FIG. 3. $\zeta_n$ on logarithmic scale with $\lambda = 2$ for example 1.

TABLE I. Errors of the numerical solutions using the maximum norm of the proposed method with $N = 13$ and $\lambda = 2$ and techniques presented in literature for example 1.

| $t$ | $e_{SM}$[37] $h = 0.001$ | $e_{ADM}$,[38] $e_{HPM}$[35] $h = 0.001$ | $e_{SM}$[35] | $e_{LWM}$[36] | $e_{SKCPs}$ |
|---|---|---|---|---|---|
| 0.0 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.2 | $1.371 \times 10^{-11}$ | 0.00 | $3.10 \times 10^{-15}$ | $1.00 \times 10^{-15}$ | $2.220\,45 \times 10^{-16}$ |
| 0.4 | $3.27 \times 10^{-11}$ | $1.00 \times 10^{-15}$ | $7.54 \times 10^{-15}$ | 0.00 | $4.440\,89 \times 10^{-16}$ |
| 0.6 | $5.86 \times 10^{-11}$ | $2.19 \times 10^{-13}$ | $1.39 \times 10^{-15}$ | $2.00 \times 10^{-15}$ | $3.541\,89 \times 10^{-16}$ |
| 0.8 | $9.54 \times 10^{-11}$ | $9.36 \times 10^{-12}$ | $2.13 \times 10^{-14}$ | $5.00 \times 10^{-15}$ | $5.420\,87 \times 10^{-16}$ |
| 1.0 | $1.43 \times 10^{-11}$ | $1.72 \times 10^{-12}$ | $3.19 \times 10^{-14}$ | $3.00 \times 10^{-15}$ | $8.881\,78 \times 10^{-16}$ |

First, we solve this equation with different values of $N$. The operational matrices of derivative and stretch are computed for $N = 3$ and $\lambda = 2$ in the following forms:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 4 & 0 & 12 & 0 \end{bmatrix}_{4\times4}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0.5 & 0 & 0 \\ 0.25 & -1 & 0.25 & 0 \\ 0.25 & 0.75 & -0.75 & 0.125 \end{bmatrix}_{4\times4},$$

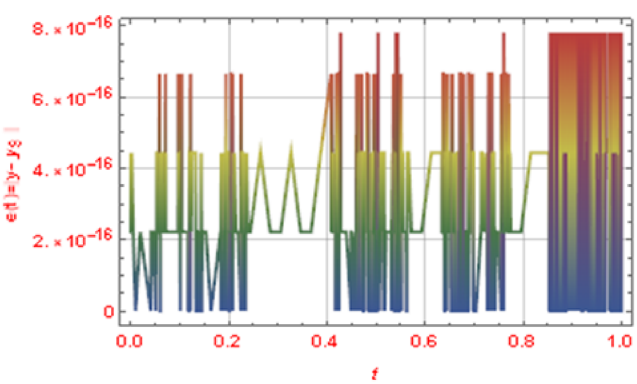


FIG. 4. Maximum norm errors for $N = 9$ and with $\lambda = 2$ for example 2.

TABLE II. Errors of the numerical solutions using the maximum norm for several values of $t$ with $\lambda = 2$ for example 2.

| $t$ | $N = 6$ | $N = 9$ |
|---|---|---|
| 0.0 | 0.000 00 | 0.000 000 |
| 0.2 | $1.221\,25 \times 10^{-14}$ | $2.220\,45 \times 10^{-16}$ |
| 0.4 | $2.242\,65 \times 10^{-14}$ | 0.000 00 |
| 0.6 | $1.865\,17 \times 10^{-14}$ | $2.220\,45 \times 10^{-16}$ |
| 0.8 | $2.775\,56 \times 10^{-14}$ | $4.440\,89 \times 10^{-16}$ |
| 1.0 | $1.998\,40 \times 10^{-13}$ | $4.440\,89 \times 10^{-16}$ |

and

$$a_0 = 1.701\,41, \quad a_1 = 0.420\,837, \quad a_2 = 0.052\,123\,3, \quad a_3 = 0.004\,026\,34$$

give

$$y_3(t) = 1 + 1.01\,043t + 0.447\,444t^2 + 0.257\,686t^3, \quad t \in [0, 1].$$

Figure 1 shows the exact and numerical solutions for $N = 12$. Figure 2 depicts the maximum norm errors by choosing $N = 12$ and $\lambda = 2$ for distinct values of $t \in [0, 1]$. In addition, Fig. 3 displays the values of $\zeta_n$ on logarithmic scale with $\lambda = 2$. Table I shows a comparison of the results for $N = 13$ and $\lambda = 2$ with the spline method (SM),[37] Adomian decomposition method (ADM),[38] homotopy perturbation method (HPM),[39] and Legendre wavelet method (LWM).[36]

TABLE III. Errors of the numerical solutions using the maximum norm for various times $t$, with $\lambda = 1.25$ for example 3.

| $t$ | $y_6$ | $y_8$ | $y_{10}$ | $y_{12}$ |
|---|---|---|---|---|
| 0.0 | 1 | 1 | 1 | 1 |
| 0.2 | 0.664 691 113 148 | 0.664 691 001 244 | 0.664 691 000 830 | 0.664 691 000 785 3 |
| 0.4 | 0.433 560 100 135 | 0.433 560 778 833 | 0.433 560 778 78 | 0.433 560 778 706 |
| 0.6 | 0.276 482 333 493 | 0.276 482 330 398 | 0.276 482 330 211 | 0.276 482 330 261 1 |
| 0.8 | 0.171 484 203 510 | 0.171 484 112 324 | 0.171 484 111 97 | 0.171 484 112 045 5 |
| 1.0 | 0.102 674 283 794 | 0.102 670 134 060 | 0.102 670 126 312 | 0.102 670 124 321 3 |

TABLE IV. Maximum norm errors of the solution for various values of $t$, with $\lambda = 1.25$ for example 3.

| $t$ | $\|y_6 - y_8\|_\infty$ | $\|y_8 - y_{10}\|_\infty$ | $\|y_{10} - y_{12}\|_\infty$ |
|---|---|---|---|
| 0.0 | $2.220\,45 \times 10^{-16}$ | 0.000 000 00 | 0.000 000 00 |
| 0.2 | $1.119\,04 \times 10^{-7}$ | $4.142\,24 \times 10^{-10}$ | $4.498\,88 \times 10^{-11}$ |
| 0.4 | $2.213\,02 \times 10^{-7}$ | $5.466\,91 \times 10^{-11}$ | $8.132\,16 \times 10^{-12}$ |
| 0.6 | $3.095\,03 \times 10^{-9}$ | $1.873\,61 \times 10^{-10}$ | $4.970\,69 \times 10^{-12}$ |
| 0.8 | $9.118\,60 \times 10^{-8}$ | $3.501\,72 \times 10^{-10}$ | $7.096\,46 \times 10^{-11}$ |
| 1.0 | $4.104\,73 \times 10^{-6}$ | $7.748\,88 \times 10^{-9}$ | $1.990\,76 \times 10^{-9}$ |

**TABLE V.** Exact and numerical solutions on the interval [0,1] for example 4.

| $t$ | Ref. 44 | Ref. 43 | Ref. 42 $M = 8$ | SKCPs $N = 8$ | SKCPs $N = 10$ | Exact |
|---|---|---|---|---|---|---|
| 0.000 | 1.000 000 00 | 0.999 999 988 | 1.000 000 000 | 1.000 000 000 | 1.000 000 000 | 1.000 000 000 |
| 0.125 | 0.855 345 1 | 0.855 345 327 6 | 0.855 345 327 2 | 0.855 345 327 30 | 0.855 345 327 304 | 0.855 345 327 304 |
| 0.250 | 0.731 611 7 | 0.731 611 628 0 | 0.731 611 629 0 | 0.731 611 628 946 | 0.731 611 628 946 | 0.731 611 628 946 |
| 0.375 | 0.625 784 0 | 0.625 784 009 8 | 0.625 784 004 9 | 0.625 784 009 60 | 0.625 784 009 604 | 0.625 784 009 645 |
| 0.500 | 0.535 246 | 0.535 246 142 8 | 0.535 246 148 5 | 0.535 246 148 51 | 0.535 246 148 519 | 0.535 246 148 519 |
| 0.625 | 0.457 833 6 | 0.457 833 261 9 | 0.457 833 261 7 | 0.457 833 261 71 | 0.457 833 261 716 | 0.457 833 261 716 |
| 0.750 | 0.391 605 4 | 0.391 605 626 2 | 0.391 605 626 7 | 0.391 605 626 76 | 0.391 605 626 768 | 0.391 605 626 768 |
| 0.875 | 0.334 958 4 | 0.334 958 042 9 | 0.334 958 042 9 | 0.334 958 042 952 | 0.334 958 042 952 | 0.334 958 042 952 |
| 1.000 | 0.286 496 5 | 0.286 504 796 8 | 0.286 504 796 8 | 0.286 504 796 8 | 0.286 504 796 8 | 0.286 504 796 8 |

*Example 2.* Next, we discuss the pantograph equation,[40]

$$\begin{cases} \dfrac{d}{dt}y(t) - y\left(\dfrac{t}{2}\right) = 0, \qquad t \in [0,1], \\ y(0) = 1, \end{cases} \tag{6.6}$$

with exact solution

$$y(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2^k(k-1)}\right)}{k!} t^k. \tag{6.7}$$

Figure 4 represents the maximum norm errors for $N = 9$ and $\lambda = 2$. Table II depicts the maximum errors for several values of $t$ with $\lambda = 2$.

*Example 3.* We consider the third example, which was proposed in Ref. 33 and extensively studied in Fox *et al.*,[41]

$$\begin{cases} \dfrac{dy(t)}{dt} = ay\left(\dfrac{t}{\lambda}\right) + by(t), \qquad t \in [0,1], \\ y(0) = y_0. \end{cases} \tag{6.8}$$

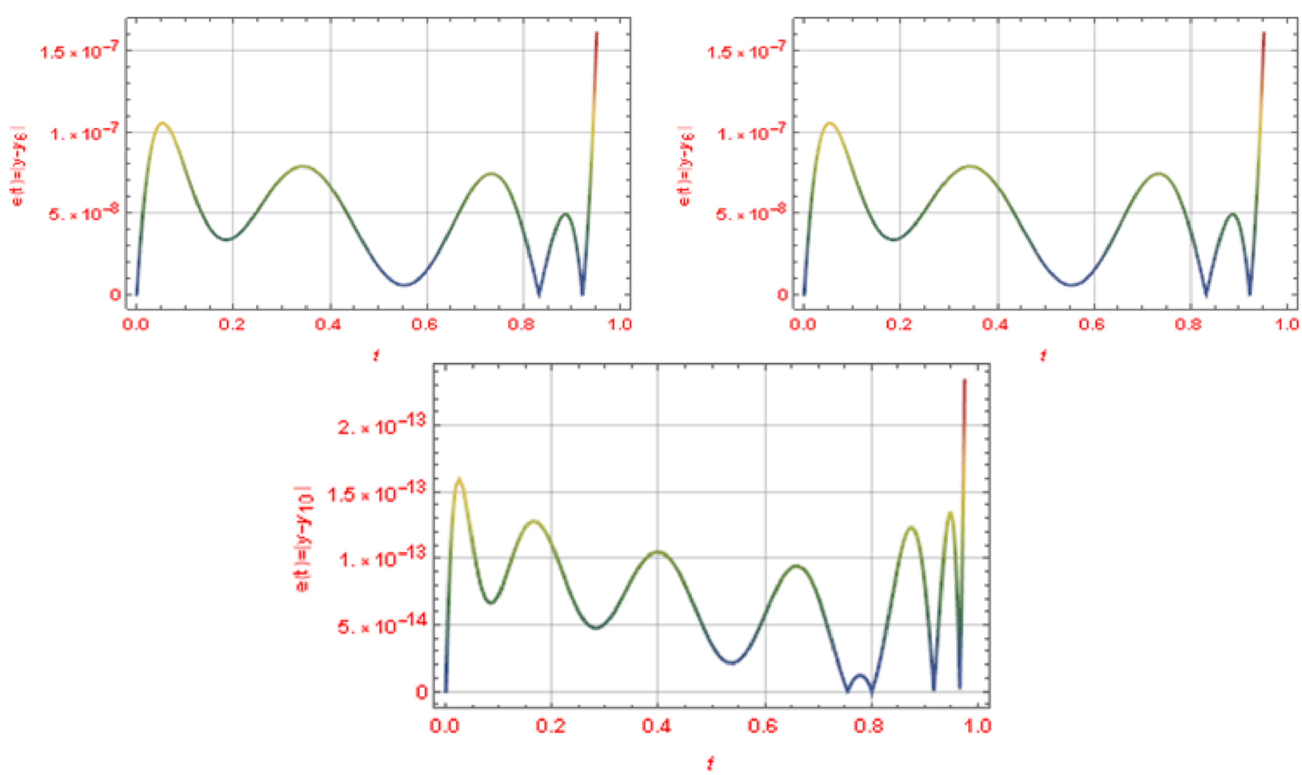


**FIG. 5.** Behavior of the maximum norm errors for $N = 6, 8, 10$, with $\lambda = \frac{5}{4}$ for example 4.

Due to the analytical solution of (6.8) not being available, many researchers have tried to approximate it. Tables III and IV report the numerical solutions and maximum norm errors for different values of $N$, with $a = -1$, $b = -1$, $\lambda = 1.25$, and $y_0 = 1$.

*Example 4.* Finally, we consider the following time-varying system, which was proposed in Refs. 42 and 43 and also extensively studied by Hsiao,[44]

$$\begin{cases} \dfrac{d}{dt}y(t) = -5e^{-t/4}y\left(\dfrac{4}{5}t\right), & t \in [0,1], \\ y(0) = 1. \end{cases} \tag{6.9}$$

Table V shows a comparison of the analytic and numerical results by using our proposed technique and other methods, such as the Haar method[44] with $M = 0.256$, Legendre wavelets[43] with $N = 4$ and $M = 6$, and the method in Ref. 44 with $M = 8$, for the values of $N = 8$ and $N = 10$ on the interval [0, 1]. We see that the errors of the proposed scheme are lesser than $10^{-10}$ for $N = 8$. Figure 5 displays the behavior of the maximum norm errors for $N = 6, 8, 10$ and $\lambda = \frac{5}{4}$.

## VII. CONCLUSIONS

This paper adopted a spectral technique based on SKCPs to approximate DDEs, which appear in various fields, including physiological processes, population dynamics, chemical kinetics, and infection evolution, to name a few. The error analysis of the spectral method was performed in detail. The numerical results from the implementation of this technique clearly showed very high accuracy compared with other known methods. The solution achieved with the present method indicated that the scheme was very efficient and useful for the numerical solution of DDEs.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Mohammad Ali Ebadi**: Investigation (equal); Methodology (equal); Software (equal); Visualization (equal); Writing – original draft (equal). **Murtadha Ali Shabeeb**: Investigation (equal); Project administration (equal); Supervision (equal). **Reza Ezzati**: Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal). **Mohammad Navaz Rasoulizadeh**: Validation (equal); Visualization (equal); Writing – original draft (equal).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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