Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (4) (2024), 1102 – 1117
DOI : 10.15672/hujms.1223815

Research Article

Transitive permutation groups with elements of movement m or $m - 2$

Mehdi Alaeiyan*, Murtadha Ali Shabeeb, Masoumeh Akbarizadeh

Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran

Abstract

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for each subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g \in G$, we define the movement of g as the max $|\Gamma^g \setminus \Gamma|$ over all subsets Γ of Ω, and the movement of G is defined as the maximum of move(g) over all non-identity elements of $g \in G$. In this paper we classify all transitive permutation groups with bounded movement equal to m that are not a 2-group, but in which every non-identity element has movement m or $m - 2$.

Mathematics Subject Classification (2020). 20B05

Keywords. permutation group, transitive, movement, fixed point free element

1. Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω. If for each subset Γ of Ω and each element $g \in G$, the size $|\Gamma^g \setminus \Gamma|$ is bounded, we define the movement of Γ as $\text{move}(\Gamma) = \max_{g\in G} |\Gamma^g \setminus \Gamma|$. Let m be a positive integer. If $\text{move}(\Gamma) \leqslant m$ for all $\Gamma \subseteq \Omega$, then G is said to have bounded movement and the movement of G is defined as the maximum of move(Γ) over all subsets Γ. This notion was introduced in [9]. Similarly, for each $1 \neq g \in G$, the movement of g is defined as the max $|\Gamma^g \setminus \Gamma|$ over all subsets Γ of Ω. If all non-identity elements of G have the same movement, then we say that G has constant movement (see [3]).

It is obvious that every permutation group in which every non-identity element moves by m or $m - 2$, is a permutation group with bounded movement equal to m. Moreover, according to Theorem 1 of [9], if G has movement equal to m, then Ω is finite, and its size is bounded by a function of m.

The intransitive permutation groups with bounded movement having maximum degree were classified in [2]. For a transitive permutation group G on a set Ω with movement m, where G is not a 2-group, the following bound on Ω was obtained in [9]. We note that for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x.

Lemma 1.1. ([9], Lemma 2.2) *Let G be a transitive permutation group on a set Ω such that G has movement equal to m. Suppose that G is not a 2-group and p is the least odd prime dividing $|G|$, then $|\Omega| \leqslant \lfloor 2mp/(p-1) \rfloor$.*

*Corresponding Author.
Email addresses: alaeiyan@iust.ac.ir (M. Alaeiyan), murtadha.alallaq@gmail.com (Murtadha A. Shabeeb), masoumeh.akbarizadeh@gmail.com (M. Akbarizadeh)
Received: 24.12.2022; Accepted: 21.09.2023

All transitive permutation groups G with bounded movement equal to m, such that G is not a 2-group but in which every non-identity element has the movement m or $m-1$ classified in [1]. There are several different kinds of transitive permutation groups that are not a 2-group and have bounded movement equal to m, but in which every non-identity element has the movement m or $m-2$. For example, it is easy to see that any non-identity member of permutation group $G=\mathbb{Z}_{4p}$ has the movement $2p$ or $2p-2$ on a set of size $n=4p$, where p is an odd prime (see Lemma 3.1).
This paper's goal is to classify all transitive permutation groups with bounded movement equal to m that are not 2-groups, but in which every non-identity element has the movement m or $m-2$.

We denote by $K \rtimes P$ a semi-direct product of K by P with normal subgroup K. The semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_2 = \langle r, s | r^n = s^2 = 1, rs = sr^{-1}\rangle$ is known as Dihedral group and it is denoted by D_{2n}.
We now have the following main theorem.

Theorem 1.2. *Let m be a positive integer and G be a transitive permutation group on a set Ω with no fixed points in Ω and also with bounded movement equal to m, in which every non-identity element has movement m or $m-2$. Moreover, suppose that G is not a 2-group and p is the least odd prime dividing $|G|$. Then G is one of the following groups:*

(1) $G \in \{S_4, \mathbb{Z}_2 \times S_4, \mathbb{Z}_2 \times A_4, (\mathbb{Z}_2)^3 \rtimes A_4, (\mathbb{Z}_2)^2 \rtimes S_4, SL(2,3)\}$, $\mid \Omega \mid = 8$ *and* $m=4$;
(2) $G := (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$, $\mid \Omega \mid = 10$ *and* $m=4$;
(3) $G \in \{S_4, S_5, A_4, A_5, \mathbb{Z}_2 \times S_4, \mathbb{Z}_2 \times S_5, \mathbb{Z}_2 \times A_4, \mathbb{Z}_2 \times A_5, (\mathbb{Z}_4)^2 \rtimes S_3, (\mathbb{Z}_2)^2 \rtimes S_4\}$, $\mid \Omega \mid = 12$ *and* $m=6$;
(4) $G \in \{F_8, A\Gamma L_1(\mathbb{F}_8)\}$, $\mid \Omega \mid = 14$ *and* $m=6$;
(5) $G \in \{S_3 \times A_5, PSL(2,17)\}$, $\mid \Omega \mid = 18$ *and* $m=8$;
(6) $G \in \{(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5, (\mathbb{Z}_2)^4 \rtimes D_{10}, \mathbb{Z}_2 \times F_8\}$, $\mid \Omega \mid = 16$ *and* $m=8$;
(7) $G := \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mid \Omega \mid = 21$ *and* $m=9$;
(8) $\in \{(\mathbb{Z}_2)^4 \rtimes D_{10}, F_5, \mathbb{Z}_2 \times F_5, \mathbb{Z}_4 \times F_5\}$, $\mid \Omega \mid = 20$ *and* $m=10$;
(9) $G \in \{(\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_3, (\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_5\}$, $\mid \Omega \mid = 25$ *and* $m=10$;
(10) $G \in \{\mathbb{Z}_{25}, D_{50}, \mathbb{Z}_5 \rtimes D_{10}, (\mathbb{Z}_5)^2 \rtimes Q_8, (\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_2, (\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_4, (\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_8, (F_5)^2\}$, $\mid \Omega \mid = 25$ *and* $m=12$;
(11) $G \in \{\mathbb{Z}_{4p}, \mathbb{Z}_4 \times D_{2p}, Dic_p = \mathbb{Z}_p \rtimes \mathbb{Z}_4, (\mathbb{Z}_2)^2 \times D_{2p}, \mathbb{Z}_2 \times \mathbb{Z}_{2p}, D_{4p}\}$, $\mid \Omega \mid = 4p$ *and* $m=2p$;
(12) $G \in \{\mathbb{Z}_4 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4, (\mathbb{Z}_2)^2 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4\}$ *where* $4 \mid p-1, \mid \Omega \mid = 4p$ *and* $m=2p$;
(13) $G \in \{\mathbb{Z}_q \rtimes \mathbb{Z}_p, AGL(1,q), \mathbb{Z}_q \rtimes \mathbb{Z}_{2p} \leqslant AGL(1,q)\}$, *where* $q=4p+1$ *is an odd prime and* $\mathbb{Z}_{2p}$ *generated by two cycles of length* $2p$, $\mid \Omega \mid = q$ *and* $m = \dfrac{q-1}{2}$;
(14) $G := K \rtimes P$, $\mid \Omega \mid = 4p$ *for* $p \geq 5$ *and* $m = 2(p-1)$, *where K is a 2-group and $P = \mathbb{Z}_p$ is fixed point free on Ω, K has p orbits of length 4 and each element of K moves at least $4(p-2)$ and at most $4(p-1)$ points of Ω.*

Note that all groups in part (2) and part (14) of Theorem 1.2, have the maximum degree mentioned in Lemma 1.1.

2. Preliminaries

Let G be a transitive permutation group on a finite set Ω. By Theorem 3.26 of [10], often known as Burnside's lemma, the average number of fixed points in Ω of elements of G is equal to the number of $G-$orbits in Ω. Since 1_G fixes $|\Omega|$ points and $|\Omega| > 1$, it follows that there is some element of G which has no fixed points in Ω. We shall say that such elements are fixed point free on Ω.
Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t non-trivial

cycles of lengths $l_1, l_2, \ldots, l_t$, say. We might represent g as

$$g = (a_1, a_2, \ldots, a_{l_1})(b_1, b_2, \ldots, b_{l_2}) \cdots (z_1, z_2, \ldots, z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \varnothing$. For example we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, \ldots, z_2, z_4, \ldots, z_{k_t}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written down. For any set $\Gamma(g)$ of this kind we say that $\Gamma(g)$ consists of *every second point of every cycle of* g. From the definition of $\Gamma(g)$, we see that

$$|\Gamma(g)^g \setminus \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \setminus \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. ([7], Lemma 2.1) *Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \setminus \Gamma| \leq \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$ where l_i is the length of the i^{th} cycle of g and t is the number of non-trivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

Remark 2.2. Let $k > 1$ be a positive integer, and let f be a cycle of length pk for some odd prime integer p. We know that f^k is k cycles of length p. We consider two cases for k.

Case 1: k is odd. Thus $k = 2t + 1$, for some $t \geq 1$. So

$$\text{move}(f) = \lfloor \frac{kp}{2} \rfloor = \lfloor \frac{(2t+1)p}{2} \rfloor = tp + \frac{p-1}{2},$$
$$\text{move}(f^k) = k\lfloor \frac{p}{2} \rfloor = (2t+1)\frac{p-1}{2} = \text{move}(f) - t.$$

Case 2: k is even. Thus $k = 2t$, for some $t \geq 1$. So

$$\text{move}(f) = \lfloor \frac{kp}{2} \rfloor = \lfloor \frac{(2t)p}{2} \rfloor = tp,$$
$$\text{move}(f^k) = k\lfloor \frac{p}{2} \rfloor = (2t)\frac{p-1}{2} = \text{move}(f) - t.$$

Therefore $\text{move}(f) - \text{move}(f^k) = \lfloor \frac{k}{2} \rfloor$.

Let m be a positive integer, and let G be a permutation group on a set Ω of size n with bounded movement equal to m, in which every non-identity element has the movement m or $m-2$. Then we have the following important result.

Proposition 2.3. *Let m be a positive integer, and let G be a permutation group on a set Ω of size n with bounded movement equal to m, in which every non-identity element has the movement m or $m-2$. Further, suppose that $1 \neq g \in G$ and $g = c_1 \cdots c_s$ is the decomposition of g into its disjoint non-trivial cycles such that $|c_i| = l_i$ for $1 \leq i \leq s$. Then one of the following holds:*

(1) *$l := l_1 = l_2 = \cdots = l_s$, where l is an odd prime or a power of 2;*
(2) *$s = 1$, such that g is one cycle of length 25;*
(3) *$s = 1$, such that g is one cycle of length $4p$, where p is an odd prime;*
(4) *$s = 2$, such that g has one cycle of length 4 and one cycle of length 5;*
(5) *$s = 2$, such that g has one cycle of length 20 and one cycle of length 5;*
(6) *$s = 2$, such that g has one cycle of length 15 and one cycle of length 3;*
(7) *$s = 2$, such that g has two cycles of length 9;*
(8) *$s = 2$, such that g has one cycle of length 9 and one cycle of length 3;*

(9) *$s = 2$, such that g has two cycles of length $2p$, where p is an odd prime;*
(10) *$s = 2$, such that g has one cycle of length 2, and and one cycle of length $2p$ where p is an odd prime;*
(11) *$s = 3$, such that g has two cycles of length 2 and one cycle of length 5;*
(12) *$s = 3$, such that g has two cycles of length 3 and one cycle of length 4;*
(13) *$s = 3$, such that g has two cycles of length 3 and one cycle of length 5;*
(14) *$s = 3$, such that g has two cycles of length 3 and one cycle of length 12;*
(15) *$s = 3$, such that g has two cycles of length 10 and one cycle of length 5;*
(16) *$s = 3$, such that g has one cycle of length 10, one cycle of length 5 and one cycle of length 2;*
(17) *$s = 4$, such that g has two cycles of length 2 and two cycles of length 3;*
(18) *$s = 4$, such that g has two cycles of length 3 and two cycles of length 6;*
(19) *$s = 4$, such that g has one cycle of length 6, two cycles of length 3 and ane cycle of length 2;*
(20) *g has one cycle of length 4 and $(s-1)$-cycles of length a power of 2 for $s \geq 2$;*
(21) *g has two cycles of length 2 and $(s-2)$-cycles of length a power of 2 for $s \geq 3$.*
Moreover, the order of g is either 6, 9, 10, 12, 15, 20, 25, p, $2p$, $4p$ or a power of 2.

Proof. Let $1 \neq g \in G$, and let $\Gamma(g)$ be the subset consisting of every second point of every cycle of g. Then by Lemma 2.1, $\text{move}(g) = \Sigma_{i=1}^{s} \lfloor l_i/2 \rfloor$. For each $1 \leq i \leq s$, we consider the element $h := g^{l_i}$ of G and compare the movement of h with the movement of g. As above, we have

$$\text{move}(h) \leq \sum_{j \neq i} \lfloor \frac{l_j}{2} \rfloor < \sum_{j=1}^{s} \lfloor \frac{l_j}{2} \rfloor = \text{move}(g).$$

We now consider the following two cases:
Case 1. Suppose $\text{move}(g) = m-2$. Then $g^{l_t} = 1$, for all $1 \leq t \leq s$. Hence $l := l_1 = l_2 = \cdots = l_s$. Suppose l is not a power of 2, and let p be an odd prime such that $l = pk$ for some positive integer k. Then by comparing the movement of g and its power g^k we obtain

$$s \lfloor \frac{l}{2} \rfloor = \text{move}(g) = \text{move}(g^k) = sk\frac{p-1}{2}.$$

It can be easily verified that $\lfloor \frac{kp}{2} \rfloor = k(p-1)/2$ if and only if $k = 1$, and so $l = p$.
Case 2. Let $\text{move}(g) = m$. Then $\text{move}(g^{l_t}) = m-2$ or $g^{l_t} = 1$, for some $1 \leq t \leq s$. Assume that there exists a $1 \leq t \leq s$ such that $\text{move}(g^{l_t}) = m-2$.
Since $g^{l_t} = c_1^{l_t} \cdots c_s^{l_t}$ and by Remark 2.2, $(l_t, l_i) = 1$, $1 \leq i \neq t \leq s$ and l_t is either $4, 5$, or $2, 3$. For the former case, g has just one cycle more than g^{l_t}. Then we can suppose that $l := l_1 = \cdots = l_{t-1} = l_{t+1} = \cdots = l_s$. Thus

$$\text{move}(g) = (s-1)\lfloor \frac{l}{2} \rfloor + 2, \quad \text{move}(g^{l_t}) = (s-1)\lfloor \frac{l}{2} \rfloor.$$

Since $\text{move}(g) - \text{move}(g^l) = 2$ and $\text{move}(g^l) = \text{move}(c_t^l)$, we have $(s-1)\lfloor \frac{l}{2} \rfloor = 2$. Hence g is either one cycle of length 4 and one cycle of length 5, two cycles of length 3 and one cycle of length 4, two cycles of length 3 and one cycle of length 5 or two cycles of length 2 and one cycle of length 5.
For the latter, g has two cycles more than g^{l_t}. As above, we can conclude that g is either two cycles of length 2 and two cycles of length 3, two cycles of length 2 and one cycle of length 5, two cycles of length 3 and one cycle of length 4 or two cycles of length 3 and one cycle of length 5.

Now suppose that $g^{l_t} = 1$, for some $1 \leq t \leq s$. Then we must have two cases:
(I) $l_i | l_t$ for all $1 \leq i \leq s$ or **(II)** $l := l_1 = l_2 = \cdots = l_s$.
In the case **(I)** we have $(l_t, l_i) \neq 1$. First we suppose that p be an odd prime and $l_t = pk$

for some positive integer $k>1$. If $(p,k)\neq 1$, then g^k is a non-identity element of G and $\text{move}(g)-\text{move}(g^k)=0$ or 2. Hence by Remark 2.2, $\text{move}(g)-\text{move}(g^k)=\lfloor\frac{k}{2}\rfloor+e$, where $e\geq 0$ is an integer. Therefore, $\lfloor\frac{k}{2}\rfloor=1$ or 2. If k is even, then $\lfloor\frac{k}{2}\rfloor=\frac{k}{2}$ implies that $k=2$ or 4, which is a contradiction. Hence $\lfloor\frac{k}{2}\rfloor=\frac{k-1}{2}=1$ or 2 implies that $k=3$ or 5, respectively. If $k=3$, then g is product of a cycle of length 3 and a cycle of length 9. If $k=5$, then g is one cycle of length 25. Now we assume that $(p,k)=1$ and g has A cycles of length k, B cycles of length p and $s-A-B$ cycles of length pk. Let c_s be the cycle of length pk. Since $\text{move}(c_s)-\text{move}(c_s^k)=\lfloor\dfrac{k}{2}\rfloor$, one has $\lfloor\dfrac{k}{2}\rfloor=1$ or 2. If $\lfloor\dfrac{k}{2}\rfloor=1$, then $k=2$ or 3. Thus $s-A-B=1$ or 2.
Let $k=2$. If $s-A-B=1$, since $\text{move}(g)-\text{move}(g^k)=A+1$ and $A\geq 0$, then $\text{move}(g)-\text{move}(g^k)=2$, so $A=1$. $\text{move}(g)-\text{move}(g^p)=2$ implies that $p=5, B=1$ or $p=3, B=2$. Therefore g is either one cycle of length 2, one cycle of length 5 and one cycle of length 10 or one cycle of length 2, two cycles of length 3 and one cycle of length 6. If $\text{move}(g)-\text{move}(g^p)=0$, then $B=0$ and g is one cycle of length 2 and one cycle of length $2p$. If $s-A-B=2$, with similar discussion as above, we have $A=0$, so $p=3, B=2$, $p=5, B=1$ or $B=0, p>3$. Hence g is either two cycles of length 3 and two cycles of length 6, one cycle of length 5 and two cycles of length 10 or two cycles of length $2p$.
Let $k=3$. So $p>3$. If $s-A-B=1$, then g is one cycle of length 3 and one cycle of length 15. If $s-A-B=2$, then $p=3$, a contradiction.
Let $k=4$. Similarly we can conclude that g is either one cycle of length 5 and one cycle of length 20 or two cycles of length 3 and one cycle of length 12.
If $k=5$, then $s-A-B=1$. So $A=0$, and $(B+1)(p-1)=4$. By $(p,k)=1$, we conclude that $B=1, p=3$. Therefore g is one cycle of length 3 and one cycle of length 15.

Now assume that $l_t=2^a$. Thus $l_i=2^{b_i}$ such that $b_i<a$. Since $g^{2^{b_i}}$ is non-identity, g is either $(s-2)$-cycles of length a power of 2 and two cycles of length 2 for $s\geq 3$, or $(s-1)$-cycles of length a power of 2 and one cycle of length 4 for $s\geq 2$. Finally, we now suppose that $l_t=2^ak$ such that $(2,k)=1$ and g has A cycles of length 2^b, B cycles of length k and $s-A-B$ cycles of length 2^ak for some integers $b<a$. Then by comparing the movement of g and its power g^k we obtain

$$\text{move}(g)=A2^{b-1}+B\lfloor\frac{k}{2}\rfloor+(s-A-B)2^{a-1}k,\ \text{move}(g^k)=A2^{b-1}+k(s-A-B)2^{a-1}.$$

If $k\geq 6$, then $\text{move}(g)-\text{move}(g^k)=B\lfloor\dfrac{k}{2}\rfloor$ implies that $\text{move}(g)-\text{move}(g^k)>2$ or $B=0$, $\text{move}(g)-\text{move}(g^{2^a})>2$, which is a contradiction. Therefore $k=3$ or 5. With similar discussion as above, we can conclude that g is either two cycles of length 2 and $(s-2)$-cycles of length a power of 2 for $s\geq 3$, or $(s-1)$-cycles of length a power of 2 and one cycle of length 4 for $s\geq 2$, two cycles of length 6 and two cycles of length 3, one cycle of length 12 and two cycles of length 3, one cycle of length 20 and one cycle of length 5 or two cycles of length 10 and one cycle of length 5.

For the case **(II)**, suppose that l is not a power of 2, and let p be an odd prime such that $l=pk$ for some positive integer k. Then we obtain that

$$\text{move}(g)=s\lfloor\frac{pk}{2}\rfloor,\quad \text{move}(g^k)=sk\frac{p-1}{2},\ \text{ and } \text{move}(g^p)=sp\lfloor\frac{k}{2}\rfloor.$$

It is straightforward to verify that $\text{move}(g^k)<m-2$ for $k\geq 6$, a contradiction. Hence we may assume that $k\leq 5$.
If $k=1$, then we have $\text{move}(g)$=$\text{move}(g^k)$ and $l=p$.
If $k=2$, then we have $\text{move}(g)$ =$\text{move}(g^p)=sp$ and $\text{move}(g^k)=s(p-1)$, this implies that $s=2$ and $l=2p$, that is, g is two cycles of length $2p$.
If $k=3$ and $p\neq 3$, then $\text{move}(g^p)<m-2$, a contradiction. Thus $p=3$. It follows that

move(g) $= 4s$ and move(g^k) =move(g^p) $= 3s$. This implies that $s = 2$ and $l = 9$, that is, g is two cycles of length 9.
If $k = 4$, then we have move(g) =move(g^p) $= 2sp$ and move(g^k) $= 2s(p-1)$, thus we must have $s = 1$ and $l = 4p$, that is, g is one cycle of length $4p$.
Finally, if $k = 5$, then move(g^p) $< m - 2$ for $p \geq 7$, a contradiction. For $p = 3$, we have move(g) $= 7s$, move(g^k) $= 5s$ and move(g^p) $= 6s$, where is a contradiction for every s. For $p = 5$, we have move(g) $= 12s$ and move(g^k) =move(g^p) $= 10s$. It follows that $s = 1$ and $l = 25$, that is, g is one cycle of length 25.

Now suppose that $l = 2k$, for some positive integer k. If k be an odd integer, then move(g) − move(g^2) $= sk - (sk - s) = s$. This implies that $s = 2$. As g^k is k cycles of length 2, we have move(g) − move(g^k) $= k(s - 1)$. So $s = k = 2$, a contradiction. Thus k is even. By comparing the movement of g and the movement of different powers of g, we obtain g is either s cycles of order $2^a, a \geq 2$, or one cycle of length $4p$ for some odd prime integer p. □

Fein et al. proved the following theorem about the finite transitive groups.

Theorem 2.4. (Fein-Kantor-Schachers theorem)[5, Theorem 1] *Let G be a finite group acting transitively on a set Ω with $|\Omega| \geq 2$. Then there exists an element of prime-power order in G acting on Ω without fixed points.*

3. Examples

Throughout this section, we assume that m is a positive integer and G is a transitive permutation group on a set Ω of size n with bounded movement equal to m, such that G is not a 2-group but in which every non-identity element has the movement m or $m - 2$. If for every $1 \neq g \in G$, move(g) $= m$, then G has constant movement which is not the purpose of this paper. So in the rest of this section we can assume that G has at least one element of movement $m - 2$.

Lemma 3.1. *The group $G = \mathbb{Z}_{4p}$ acts transitively on a set of size $n = 4p$, where p is an odd prime, and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let $1 \neq g \in G$. Then it can be easily shown that g has order 2, 4, p, $2p$ or $4p$. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. If $o(g) = 2$, then g has $2p$ cycles of length 2 and hence $|\Gamma(g)^g \setminus \Gamma(g)| = 2p$, that is, move($g$) $= 2p$. If $o(g) = 4$, then g has p cycles of length 4 and hence $|\Gamma(g)^g \setminus \Gamma(g)| = 2p$, that is, move($g$) $= 2p$. If $o(g) = p$, then g has 4 cycles of length p and hence $|\Gamma(g)^g \setminus \Gamma(g)| = 4\frac{p-1}{2} = 2p - 2$, that is, move($g$) $= 2p - 2$. Finally, if $o(g) = 2p$ or $4p$ then g has 2 cycles of length $2p$ and a cycle of length $4p$, respectively. As above it is easy to see that move(g) $= 2p$. It follows that every non-identity element of G has movement $2p$ or $2p - 2$. □

Lemma 3.2. *The group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2p}$ acts transitively on a set of size $n = 4p$, where p is an odd prime, and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let $1 \neq g \in G$. Then g is either 2 cycles of length $2p$, 4 cycles of length p or $2p$ cycles of length 2. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. Therefore $|\Gamma(g)^g \setminus \Gamma(g)| = 2p, 2p - 2$, or $2p$, respectively. This implies that every non-identity element of G has movement $2p$ or $2p - 2$. □

Let H be cyclic of order n and $K = \langle k \rangle$ be cyclic of order m and suppose r is an integer such that $r^m \equiv 1 (\mathrm{mod}\ n)$. For $i = 1, ..., m$, let $(k^i)\theta : H \to H$ be defined by $h^{(k^i)\theta} = h^{r^i}$ for h in H. It is straightforward to verify that each $(k^i)\theta$ is an automorphism of H, and that θ is a homomorphism from K to Aut(H). Hence the semi-direct product $G = H \rtimes K$ (with respect to θ) exists and if $H = \langle h \rangle$, then G is given by the defining relations:

$$h^n = 1, \quad k^m = 1, \quad k^{-1}hk = h^r, \quad \text{with} \quad r^m \equiv 1 (\mathrm{mod}\ n).$$

Here every element of G is uniquely expressible as $h^i k^j$, where $0 \leq i \leq n-1,\ 0 \leq j \leq m-1$. Certain semi-direct products of this type (as a permutation group on a set Ω of size n) also provide examples of transitive permutation groups where every non-identity element has the movement m or $m-2$, and the bound in Lemma 1.1 is not attained (as the following lemma shows). We note that, if $n = q$, a prime, then by Theorem 3.6.1 of [11], this group G is a subgroup of the Frobenius group $AGL(1, q) = \mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}$.

Lemma 3.3. *Let $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}$ be the group defined as above of order $q(q-1)$, where $q := 4p + 1$ is an odd prime. Then G acts transitively on a set of size $n = q$ and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. By the above statement, the group G is a Frobenius group and has up to permutational isomorphism a unique transitive representation of degree q on a set Ω. Let $g \in G, \mathrm{o}(g) = q$. If $\Gamma(g)$ consists of every second point of the unique cycle of g, then $\mathrm{move}(g) = \frac{q-1}{2} = 2p$. Since the order of each element of G is either $2, 4, p, q, 2p$ or $4p$, so by Lemma 3.1, every non-identity element has movement $2p$ or $2p - 2$. □

Corollary 3.4. *Let $G \leqslant AGL(1, q)$ be a semi-direct product $\mathbb{Z}_q \rtimes \mathbb{Z}_{2p}$, where p, $q = 4p+1$ are odd primes and $\mathbb{Z}_{2p}$ generated by two cycles of length $2p$. Then G acts transitively on a set of size $n = q$ and in this action every non-identity element has movement $2p$ or $2p-2$.*

Lemma 3.5. *Let G be a semi-direct product $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Then G acts transitively on a set of size $n = 21$ and in this action every non-identity element has movement 7 or 9.*

Proof. Let G be a semi-direct product $G = \mathbb{Z}_q \rtimes \mathbb{Z}_p$, where p, q are odd primes and $q = 2p + 1$. Then as an immediate consequence of the above statement of Lemma 3.3, G acts transitively on a set Ω of size pq whenever $p \mid (q-1)$. Let $1 \neq g \in G$. So the order of g is either p or q. If $\mathrm{o}(g) = q$, then g is p cycles of length q. Hence $\mathrm{move}(g) = p\dfrac{q-1}{2} = p^2$. If $\mathrm{o}(g) = p$, then g is q cycles of length p. Hence $\mathrm{move}(g) = q\dfrac{p-1}{2}$. These are hold if and only if $p = 3$, since every non-identity element has the movement m or $m - 2$. It follows that $m = 9$. □

Lemma 3.6. *Let G be a semi-direct product $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, where $p, q = 4p + 1$ are prime integers. Then G acts transitively on a set of size $n = q$ and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let G be a semi-direct product $G = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ where, p, q are odd primes and $q = 4p+1$, then G acts transitively on a set Ω of size q. Let $1 \neq g \in G$. Then the order of g is either p or q. If $\mathrm{o}(g) = q$, then g is a cycle of length q. So $\mathrm{move}(g) = \dfrac{q-1}{2} = 2p$. If $\mathrm{o}(g) = p$, then g is a product of 4 disjoint cycles of length p. So $\mathrm{move}(g) = 4\dfrac{p-1}{2} = 2p-2$. Thus every non-identity element of G has movement $2p$ or $2p - 2$. □

Lemma 3.7. *The groups $\mathbb{Z}_{25}$ and $G = D_{50}$ act transitively on a set of size $n = 25$ and in this action every non-identity element has movement 12 or 10.*

Proof. Let $M := \langle \alpha \rangle$ and $N := \langle \beta \rangle$ be two cyclic permutation groups on the set $\Omega = \{1, 2, \ldots, 25\}$, where $\alpha = (1\ 2\ \cdots\ 25)$ and $\beta = (1\ 3)(4\ 25)(5\ 24)\cdots(14\ 15)$. It is straightforward to verify that $M \cong \mathbb{Z}_{25}$ and $D_{50} \cong \langle M, N \rangle$. Since $M \leqslant G$ acts transitively on a set Ω, so G is a transitive permutation group on a set Ω. Let $1 \neq g \in M$, then it is easy to see that g has order 5 or 25. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. If $\mathrm{o}(g) = 25$ then g is a cycle of length 25 and hence $|\Gamma(g)^g \setminus \Gamma(g)| = 12$, that is, $\mathrm{move}(g) = 12$. Now, if $\mathrm{o}(g) = 5$ then g has 5 cycles of length 5 and hence $|\Gamma(g)^g \setminus \Gamma(g)| = 5\lfloor \frac{5}{2} \rfloor = 10$, that is, $\mathrm{move}(g) = 10$. Let $1 \neq g \in \langle M, N \rangle$, $g \notin M$ and $g \notin N$. Then g has 12 cycles of length 2 and similarly, $\mathrm{move}(g) = 12$. This implies that every non-identity element of G has movement 12 or 10. □

Lemma 3.8. *The group $G = D_{2n}$, where $n = 2p$, acts transitively on a set of size $4p$ and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let $\mathbb{Z}_{2p} := \langle (1\ 2\ ...\ 2p)(1'\ 2'\ ...\ 2p')\rangle$ and $\mathbb{Z}_2 := \langle (1\ 1')(2\ 2')...(2p\ 2p')\rangle$ be two cyclic permutation groups on the set $\Omega = \{1, 2, ..., 2p, 1', 2', ..., 2p'\}$. Then it can be easily shown that the group $G = D_{4p} \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_2$ acts transitively on a set Ω of size $4p$ and in this action every non-identity element of G has movement $2p$ or $2p - 2$. □

Lemma 3.9. *Let p be an odd prime.The Dicyclic group $\mathrm{Dic}_p = \mathbb{Z}_p \rtimes \mathbb{Z}_4$ acts transitively on a set of size $4p$ and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let a be a positive integer and p be an odd prime. Then the Frobenius group $G = \mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$ acts transitively on a set Ω of size $p2^a$. Every non-identity element of $g \in G$ has order 2, p or $2^i(1 < i \leq a)$. Hence $\mathrm{move}(g) = p2^{a-1}\lfloor \frac{2}{2} \rfloor = p2^{a-1}$, $2^a\lfloor \frac{p}{2} \rfloor = 2^{a-1}p - 2^{a-1}$ or $p2^{a-i}\lfloor \frac{2^i}{2} \rfloor = 2^{a-1}p$. Therefore, $a = 2$ and $m = 2p$. □

Lemma 3.10. *Let p be an odd prime. Then The group $G = \mathbb{Z}_4 \times D_{2p}$ acts transitively on a set Ω of size $n = 4p$, and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let $1 \neq g \in G$. Then g is either one cycle of length $4p$, two cycles of length $2p$, four cycles of length p, p cycles of length 4, $2p - 2$ cycles of length 2 or $2p$ cycles of length 2. Therefore every non-identity element of G has movement $2p$ or $2p - 2$. □

Lemma 3.11. *Let p be an odd prime. Then The group $G = (\mathbb{Z}_2)^2 \times D_{2p}$ acts transitively on a set Ω of size $n = 4p$, and in this action every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Let $1 \neq g \in G$. Then g is either two cycles of length $2p$, four cycles of length p, $2p - 2$ cycles of length 2 or $2p$ cycles of length 2. Therefore every non-identity element of G has movement $2p$ or $2p - 2$. □

Lemma 3.12. *Let p be an odd prime such that $4|p - 1$. Then $G = \mathbb{Z}_4 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4$ and $T = (\mathbb{Z}_2)^2 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4$ act transitively on a set of size $4p$ and every non-identity element has movement $2p$ or $2p - 2$.*

Proof. Since $4|p - 1$, $\mathbb{Z}_p \rtimes \mathbb{Z}_4$ acts transitively on a set of size p. Hence G and T are transitive groups on $4p$ points. Let $1 \neq g \in G$. Then g is either $2p$ cycles of length 2, $(2p-2)$ cycles of length 2, p cycles of length 4, $(p-1)$ cycles of length 4, 2 cycles of length 2 and $(p - 1)$ cycles of length 4, 4 cycles of length p, two cycles of length $2p$, or one cycle of length $4p$.
Every non-identity element $t \in T$ is either $2p$ cycles of length 2, $(2p - 2)$ cycles of length 2, $(p - 1)$ cycles of length 4, 2 cycles of length 2 and $(p - 1)$ cycles of length 4, 4 cycles of length p, or two cycles of length $2p$. Therefore every non-identity element of G and T has movement $2p$ or $2p - 2$. □

Let $N := \langle (i\ i')|i = 1, 2, ..., p\rangle$ be a permutation group of degree $2p$ on the set

$$\Omega = \{1, 1', 2, 2', ..., p, p'\}.$$

Moreover, suppose that

$$M := (\mathbb{Z}_2)^{p-1} = \langle z_i = (i\ i')(i + 1\ i' + 1)|1 \leq i \leq p - 1\rangle,$$

is the subgroup of N of even permutations in N. Set

$$g = (12...p)(1'2'...p').$$

Then g normalizes M and we consider the permutation group

$$G := M \rtimes \mathbb{Z}_p = \langle z_1, z_2, ..., z_{p-1}\rangle \rtimes \langle g\rangle$$

on Ω. Now $z_i^g = z_{i+1}$, for $1 \leq i < p-1$, and $z_{p-1}^g = z_1 z_2 ... z_{p-1}$. This group G provides an example of transitive permutation group in which every non-identity element has the movement m or $m-2$, and the bound in Lemma 1.1 is attained.
As the group M is a 2-group, so by definition there is an element of movement equal to 2 in M. Also, the group $\mathbb{Z}_p = \langle g\rangle$ has constant movement equal to $p-1$. Now, if every non-identity element of G has movement $m = p-1$ or $m-2$, then $m = 4$ and $p = 5$. Consequently, the group $G = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$ acts transitively on a set of size 10, and in this action every non-identity element has movement 4 or 2. Therefore, we can conclude the following lemma.

Lemma 3.13. *Let G be a semi-direct product $G = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$, as above. Then G acts transitively on a set of size $n = 10$ and in this action every non-identity element has movement 2 or 4.*

Example 3.14. Let $G = (\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_k$ acts transitively on a set Ω of size 25, where $k \in \{3, 5\}$ and in this action every non-identity element has movement 8 or 10.

Proof. Let $1 \neq g \in G$. If $k = 3$, then g has order 3 or 5. Therefore, move$(g) = 8\lfloor\frac{3}{2}\rfloor = 8$ or $5\lfloor\frac{5}{2}\rfloor = 10$. If $k = 5$, then the order of g is 5 and move$(g) = 5\lfloor\frac{5}{2}\rfloor = 10$, or $4\lfloor\frac{5}{2}\rfloor = 8$. □

Example 3.15. In [4] the transitive groups of degree up to 31 has been listed. So, we know that there are more transitive groups on 25 points. By using Gap [6], we list the ones in which every non-identity element has movement m or $m-2$ in Table 1.

Table 1. Transitive action on 25 points

Number of points $= 25$	
Group: $\mathbb{Z}_5 \rtimes \mathbf{D}_{10}$	**Movement**$= 12$
Element Description	**Movement of Elements**
12 cycles of length 2	12
5 cycles of length 5	10
Group: $(\mathbb{Z}_5)^2 \rtimes \mathbf{Q}_8$	**Movement**$= 12$
Element Description	**Movement of Elements**
12 cycles of length 2	12
6 cycles of length 4	12
5 cycles of length 5	10
Group: $(\mathbb{Z}_5)^2 \rtimes (\mathbb{Z}_4)^2$	**Movement**$= 12$
Element Description	**Movement of Elements**
12 cycles of length 2	12
10 cycles of length 2	10
6 cycles of length 4	12
5 cycles of length 4	10
2 cycles of length 2 and 5 cycles of length 4	12
5 cycles of length 5	10
one cycle of length 5 and 2 cycles of length 10	12
one cycle of length 5 and one cycle of length 20	12
Group: $(\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_2$	**Movement**$= 12$
Element Description	**Movement of Elements**
10 cycles of length 2	10
5 cycles of length 5	10
one cycle of length 5 and 2 cycles of length 10	12
Group: $(\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_4$	**Movement**$= 12$
Element Description	**Movement of Elements**
12 cycles of length 2	12
6 cycles of length 4	12
5 cycles of length 5	10
Group: $(\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_8$	**Movement**$= 12$
Element Description	**Movement of Elements**
12 cycles of length 2	12
6 cycles of length 4	12
5 cycles of length 5	10
3 cycles of length 8	12

Example 3.16. Let $G \in \{\mathbb{Z}_2 \times \mathrm{S}_4,\ \mathbb{Z}_2 \times \mathrm{A}_4\}$. If G acts transitively on a set of size $n = 8$, then every non-identity element has movement 4 or 2.

Proof. Let $1 \neq g \in G$. If $G = \mathbb{Z}_2 \times \mathrm{S}_4$, then g is either a product of 4 disjoint cycles of length 2 or 2 disjoint cycles of length 2, a product of 2 disjoint cycles of length 3, a product of 2 disjoint cycles of length 4, a product of two disjoint cycles, one cycle of length 2 and one cycle of length 6. If $G = \mathbb{Z}_2 \times \mathrm{A}_4$, then g is either a product of 4 disjoint cycles of length 2, a product of 2 disjoint cycles of length 3, a product of two disjoint cycles, one cycle of length 2 and one cycle of length 6. Therefore, every non-identity element of G has movement 4 or 2. □

Example 3.17. Let G be a transitive group on 8 points which every non-identity element has movement m or $m-2$. By using [4] and Gap [6], all these groups and their elements are described in Table 2.

Table 2. Transitive action on 8 points

Number of points $= 8$	
Group: $\mathbf{S}_4$	**Movement**$= 4$
Element Description	**Movement of Elements**
4 cycles of length 2	4
2 cycles of length 3	2
2 cycles of length 4	4
Group: $\mathbf{SL}_2(3)$	**Movement**$= 4$
Element Description	**Movement of Elements**
4 cycles of length 2	4
2 cycles of length 3	2
2 cycles of length 4	4
one cycle of length 2 and one cycle of length 6	4
Group: $(\mathbb{Z}_2)^3 \rtimes \mathbf{A}_4$	**Movement**$= 4$
Element Description	**Movement of Elements**
4 cycles of length 2	4
2 cycles of length 2	2
2 cycle of length 3	2
2 cycles of length 4	4
one cycle of length 2 and one cycle of length 6	4
Group: $(\mathbb{Z}_2)^2 \rtimes \mathbf{S}_4$	**Movement**$= 4$
Element Description	**Movement of Elements**
4 cycles of length 2	4
2 cycles of length 2	2
2 cycle of length 3	2
2 cycles of length 4	4

Example 3.18. From [4], all transitive groups on 12 points were determined. By using Gap[6], we list the cases in which every non-identity element has movement 6 or 4 In the Table 3.

Table 3. Transitive action on 12 points

Number of points = 12	
Group: $\mathbf{S}_4$	**Movement**= 6
Element Description	**Movement of Elements**
4 cycles of length 2	4
6 cycles of length 2	6
4 cycles of length 3	4
2 cycles of length 2 and 2 cycles of length 4	6
Group: $\mathbf{A}_4$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycles of length 2	6
4 cycles of length 3	4
Group: $\mathbf{S}_5$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycle of length 2	6
4 cycles of length 2	4
4 cycles of length 3	4
2 cycle of length 2 and 2 cycles of length 4	6
2 cycles of length 5	4
2 cycles of length 6	6
Group: $\mathbf{A}_5$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycle of length 2	6
4 cycles of length 3	4
2 cycles of length 5	4
Group: $(\mathbb{Z}_4)^2 \rtimes \mathbf{S}_3$	**Movement**= 6
Element Description	**Movement of Elements**
4 cycles of length 2	4
4 cycle of length 3	4
2 cycles of length 4	4
2 cycles of length 2 and 2 cycles of length 4	6
one cycle of length 4 and one cycle of length 8	6
Group: $(\mathbb{Z}_2)^2 \rtimes \mathbf{S}_4$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycles of length 2	6
4 cycles of length 2	4
4 cycle of length 3	4
2 cycles of length 2 and 2 cycles of length 4	6
Group: $\mathbb{Z}_2 \times \mathbf{S}_4$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycles of length 2	6
4 cycles of length 2	4
4 cycle of length 3	4
2 cycles of length 2 and 2 cycles of length 4	6
2 cycle of length 6	6
6 cycles of length 2	6
4 cycles of length 2	4
4 cycle of length 3	4
2 cycles of length 2 and 2 cycles of length 4	6
2 cycle of length 4	4
2 cycle of length 5	4
2 cycle of length 6	6
one cycle of length 2 and one cycle of length 10	6
Group: $\mathbb{Z}_2 \times \mathbf{A}_4$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycles of length 2	6
4 cycles of length 2	4
4 cycle of length 3	4
2 cycle of length 6	6
Group: $\mathbb{Z}_2 \times \mathbf{A}_5$	**Movement**= 6
Element Description	**Movement of Elements**
6 cycles of length 2	6
4 cycles of length 2	4
4 cycle of length 3	4
2 cycle of length 5	4
2 cycle of length 6	6
one cycle of length 2 and one cycle of length 10	6

Example 3.19. From [4], all transitive groups on $14, 16$ and 18 points were determined. By using Gap[6], we list the cases in which every non-identity element has movement m or $m-2$ in the Tables 4, 5 and 6, respectively.

Table 4. Transitive action on 14 points

Number of points $= 14$	
Group: $\mathbf{A\Gamma L}_1(\mathbb{F}_8)$	**Movement**$= 6$
Element Description	**Movement of Elements**
4 cycles of length 2	4
4 cycles of length 3	4
2 cycles of length 7	6
one cycle of length 2 and two cycles of length 3 and one cycle of length 6	6
Group: $(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_7 = \mathbb{F}_8$	**Movement**$= 6$
Element Description	**Movement of Elements**
4 cycles of length 2	4
2 cycles of length 7	6

Table 5. Transitive action on 16 points

Number of points $= 16$	
Group: $(\mathbb{Z}_2)^4 \rtimes \mathbf{D}_{10}$	**Movement**$= 8$
Element Description	**Movement of Elements**
8 cycles of length 2	8
6 cycles of length 2	6
4 cycles of length 4	8
3 cycles of length 5	6
Group: $\mathbb{Z}_2 \times \mathbf{F}_8$ $(\mathbf{F}_8 = (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_7)$	**Movement**$= 8$
Element Description	**Movement of Elements**
8 cycles of length 2	8
2 cycles of length 7	6
one cycle of length 2 and one cycle of length 14	8
Group: $(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$	**Movement**$= 8$
Element Description	**Movement of Elements**
8 cycles of length 2	8
3 cycles of length 5	6

Table 6. Transitive action on 18 points

Number of points $= 18$	
Group: $\mathbf{S}_3 \times \mathbf{A}_5$	**Movement**$= 8$
Element Description	**Movement of Elements**
8 cycles of length 2	8
6 cycles of length 2	6
6 cycles of length 3	6
3 cycles of length 5	6
2 cycles of length 3 and 2 cycles of length 6	8
One cycle of length 2 and One cycle of length 5 and One cycle of length 10	8
One cycle of length 3 and One cycle of length 15	8
Group: $\mathbf{PSL}(2, 17)$	**Movement**$= 8$
Element Description	**Movement of Elements**
8 cycles of length 2	8
6 cycles of length 3	6
4 cycles of length 4	8
2 cycles of length 8	8
2 cycles of length 9	8
One cycle of length 17	8

Example 3.20. By Lemma 3.9, [4] and Gap [6], we can describe all transitive group on 20 points which every non-identity element has movement m or $m-2$ in the Table 7.

Table 7. Transitive action on 20 points

Number of points = 20	
Group: $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	**Movement**= 10
Element Description	**Movement of Elements**
10 cycles of length 2	10
5 cycles of length 4	10
4 cycles of length 5	8
Group: $\mathbb{Z}_2 \times \mathbf{F}_5$ ($\mathbf{F}_5 := \mathbb{Z}_5 \rtimes \mathbb{Z}_4$)	**Movement**= 10
Element Description	**Movement of Elements**
10 cycles of length 2	10
8 cycles of length 2	8
5 cycles of length 4	10
4 cycles of length 5	8
2 cycles of length 10	10
Group: $\mathbb{Z}_4 \times \mathbf{F}_5$ ($\mathbf{F}_5 := \mathbb{Z}_5 \rtimes \mathbb{Z}_4$)	**Movement**= 10
Element Description	**Movement of Elements**
10 cycles of length 2	10
8 cycles of length 2	8
5 cycles of length 4	10
4 cycles of length 4	8
2 cycles of length 2 and 4 cycles of length 4	10
4 cycles of length 5	8
2 cycles of length 10	10
one cycle of length 20	10
Group: $(\mathbb{Z}_2)^4 \rtimes \mathbf{D}_{10}$	**Movement**= 10
Element Description	**Movement of Elements**
8 cycles of length 2	8
2 cycles of length 2 and 4 cycles of length 4	10
4 cycles of length 5	8

4. Proof of Theorem 1.2

Now, we are ready to complete the proof of Theorem 1.2:
Let G, Ω and m be as in Theorem 1.2, with $n := |\Omega|$ and move$(G) = m$. We consider the following two possibilities:
Case 1: n is the maximum possible degree as in Lemma 1.1.
A transitive permutation group of degree $3m$ (which is the bound of Lemma 1.1, for $p = 3$) with bounded movement equal to m, were classified in [8] and the examples are as follows:
(a) $G := S_3,\ m = 1$;
(b) $G := A_4$ or $A_5,\ m = 2$;
(c) G is a 3-group of exponent 3.
It can be easily verified that the movement of all of these groups are not m or $m-2$, which is a contradiction.
But for $p \geq 5$, by Theorem 1.2 of [7], one of the following holds:
(1) $|\Omega| = p,\ m = (p-1)/2$ and $G := \mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$, where $2^a|(p-1)$ for some $a \geq 1$.
(2) $|\Omega| = 2^s p,\ m = 2^{s-1}(p-1),\ 1 < 2^s < p$, and $G := K \rtimes P$ with K a 2-group and $P = \mathbb{Z}_p$ is fixed point free on Ω; K has p-orbits of length 2^s, and each element of K moves at most $2^s(p-1)$ points of Ω.
(3) G is a p-group of exponent bounded in terms of p only.
By Theorem 1.1 of [3], all groups in part (1) and part (3) are examples in which every non-identity element has the same movement equal to m. In part (2), suppose that each element of K moves at least $2^s k$ points of Ω, where $k \leq p-1$ is an integer. Now, if every non-identity element of G has movement $m = 2^{s-1}(p-1)$ or $m-2 = 2^{s-1}k$, then $2^{s-1}(p-1-k) = 2$. Hence $s \leq 2$. According to Lemma 3.13, we have $G = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$ for $s = 1$. If $s = 2$, then we have $k = p-2$ but we do not know any examples.
Case 2: n is not the maximum possible degree as in Lemma 1.1.
Let $1 \neq g \in G$. Then by Proposition 2.3, g in its disjoint cycle representation has either one cycle of length $4p$, one cycle of length 25, one cycle of length 4 and one cycle of length 5, two cycles of length 9, one cycle of length 20 and one cycle of length 5, two cycles of length $2p$, one cycle of length 2 and one cycle of length $2p$, two cycles of length 2 and one

cycle of length 5, two cycles of length 3 and one cycle of length 4, two cycles of length 3 and one cycle of length 5, two cycles of length 3 and one cycle of length 12, two cycles of length 10 and one cycle of length 5, two cycles of length 2 and two cycles of length 3, two cycles of length 3 and two cycles of length 6, one cycle of length 15 and one cycle of length 3, the disjoint product of three cycles, one cycle of length 2, one cycle of length 5 and one cycle of length 10, the disjoint product of four cycles, one cycle of length 2, two cycles of length 3 and one cycle of length 6, $(s-1)-$cycles of length a power of 2 and one cycle of length 4 for $s \geq 2$, $(s-2)-$cycles of length a power of 2 and two cycles of length 2 for $s \geq 3$, multiple cycles of length q (for some prime q) or multiple cycles of length a power of 2, say g_{4p}, g_{25}, $g_{4,5}$, $g_{9,9}$, $g_{20,5}$, $g_{2p,2p}$, $g_{2,2p}$, $g_{2,2,5}$, $g_{3,3,4}$, $g_{3,3,5}$, $g_{3,3,12}$, $g_{10,10,5}$, $g_{2,3,2,3}$, $g_{3,6,3,6}$, $g_{3,15}$, $g_{2,5,10}$, $g_{3,2,3,6}$, $g^*_{2^a,4}$, $g^*_{2^a,2,2}$, g^*_q, $g^*_{2^a}$, respectively.

Let G be a transitive permutation group on a set Ω and move$(G) = m$. By definition of move(G) and Proposition 2.3, we have,

$m \in \{2, 4, 6, 8, 12, 2p, p+1, 2+(s-1)2^{a-1}, 2+(s-2)2^{a-1}, \frac{s(q-1)}{2}, s2^{a-1}\}$.

First suppose that $m = 12$. Then $g_{25}, g_{20,5}, g_{10,10,5}, g^*_3(s = 12)$, $g^*_2(s = 12)$ or $g^*_{4,2,2}(s = 7)$ could belong to G. If $g^*_3(s = 12) \in G$, sine G is transitive, then G must have an element whose form is a cycle of length 36, say $g^{'}$, hence move$(g^{'}) = 18$, which is a contradiction. Hence from Lemma 1.1, $|\ \Omega\ | \leqslant \lfloor 2 \times 12 \times 5/(5-1) \rfloor = 30$. Therefore, by Lemma 3.7 and Example 3.15 $G = \mathbb{Z}_{25}, D_{50}$ or G is one of groups listed in Table 1.

Let $m = 10$. Then $g_{20}, g_2 * (s = 10), g_3 * (s = 10), g_{10,10}, g^*_4(s = 5), g^*_{4,2,2}(s = 6)$ could belong to G. Thus by Lemma 1.1 $|\ \Omega\ |$ is at most 30. Therefore, by Lemmas 3.1, 3.8, and 3.10, Examples 3.14 and 3.20 G is $\mathbb{Z}_{20}$, $\mathrm{D}_{2n}, n = 10$, $\mathbb{Z}_4 \times \mathrm{D}_{10}$, $(\mathbb{Z}_5)^2 \rtimes \mathbb{Z}_k$ where $k = 3, 5$ and one of groups listed in Table 7, respectively.

Let $m = 8$. Then $g_{9,9}, g_{3,3,12}, g_{3,6,3,6}, g_{3,15}, g_{2,5,10}, g_{2,14}$ could belong to G. Sine the least odd prime dividing $|G|$ is either $3, 5$ or 7, by lemma 1.1, $|\ \Omega\ |$ is at most 24. Thus by [4] and [6], G is one of the groups listed in Tables 5 and 6.

Let $m = 6$. Then $g_{6,6}, g_{12}, g_{2,10}, g_{3,2,3,6}, g_{2,2,4,4}, g_{4,8}, g^*_2(s = 6), g^*_7(s = 2), g^*_{13}(s = 1)$ could belong to G. Hence G is a transitive group on $12, 13$ or 14 points and the least odd prime dividing $|G|$ is 3 or 5. So by lemma 1.1, $|\ \Omega\ | \leqslant 18$. Therefore by Lemmas 3.1 - 3.10 and Examples 3.18 and 3.19, G is $\mathbb{Z}_{12}, \mathbb{Z}_{13} \rtimes \mathbb{Z}_{12}, \mathbb{Z}_{13} \rtimes \mathbb{Z}_6, \mathbb{Z}_{13} \rtimes \mathbb{Z}_3, \mathrm{D}_{12}, \mathbb{Z}_4 \times \mathrm{D}_6$ or one of the groups listed in Tables 3 and 4.

Let $m = 5$. Then $g_{9,3} \in G$ and $|\ \Omega\ | \leqslant 15$. However, by [4] there is not such a group G.

Let $m = 4$. Then $g_{4,4}, g_{4,5}, g_{2,6}, g_{2,2,5}, g_{3,3,4}, g_{3,3,5}, g_{2,2,3,3}, g^*_{4,2,2}(s = 3), g^*_2(s = 4), g^*_5(s = 2)$ could belong to G. If $g_{4,5}, g_{2,2,5} \in G$, since G is transitive, then by Theorem 2.4 G must have an element whose form is a cycle of length 9, say $\widehat{g}$, hence move$(\widehat{g}^3) = 3$, which is a contradiction. Therefore G is either transitive on 10 points and by Lemma 3.13, $G = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$, or G is transitive on 8 points and G is one of the groups in Examples 3.16 and 3.17.

If $m = s2^k$ $(s \geq 1)$ and G consists precisely of the elements of the form $g^*_{2^a}$, then G is a 2-group, which is not included in our classification. Since every non-identity element of G has the movement m or $m-2$, s must be a prime integer and $|\ \Omega\ |= 2^{k+1}s$. Therefore by case 1 (2), $k = 0, s = 5$ and $G = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$.

Let $m = \frac{s(q-1)}{2}$ and $g^*_q \in G$, then G is transitive on sq points. Let G is not a q-group. If G has an element whose form is a cycle of length sq, say $g^{'}$, then $(g^{'})^q$ is a product of q disjoint cycles of length s, and move$((g^{'})^q) = q\lfloor\frac{s}{2}\rfloor$. If s is even, then $|\text{move}(g^{'}) - \text{move}((g^{'})^q)| = \frac{s}{2}$. Thus $s = 4$ and by Lemma 4.7 of [3], n is the maximum possible degree and the groups satisfying in this case are those mentioned in Case 1. If s is odd, then $|\text{move}(g^{'}) - \text{move}((g^{'})^q)| = |\text{move}(g^{'}) - \text{move}((g^{'})^s)| = 2$ implies that $s = p = 5$, which is a contradiction. Let $g^{'} \notin G$ and $g^*_q, g^*_s \in G$. Then by Lemmas 3.5 and 3.6,

$G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ where $p, q = 4p+1$ are prime integers.
Let $m = 2p$. If $g_{4p} \in G$, then with this guess that G is a cyclic group we have $G = \mathbb{Z}_{4p}$. Otherwise, G must consists precisely of those elements whose forms are $g_{2p,2p}$, $g_p^*(s=4)$ and $g_2^*(s=2p)$ and it also has a subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_p$. Hence by Lemmas 3.10 and 3.12 $G = \mathbb{Z}_4 \times \mathrm{D}_{2p}$ and $\mathbb{Z}_4 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4$, respectively.
If $g_q^* \in G$, where $q = 4p+1$ is prime, then G is transitive on $4p+1$ points and by Lemma 3.3, Corollary 3.4 and Lemma 3.6, we have $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}, \mathbb{Z}_q \rtimes \mathbb{Z}_{2p}, \mathbb{Z}_q \rtimes \mathbb{Z}_p \ (4 \mid p-1)$, respectively. If $g_{4p}, g_{4p+1} \notin G$, then $g_{2p,2p} \in G$ and by Lemmas 3.2, 3.8, 3.9, 3.11 and 3.12 $G = \mathbb{Z}_2 \times \mathbb{Z}_{2p}$, $\mathrm{D}_{2n}\ (n=2p)$, $\mathbb{Z}_p \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_2)^2 \times \mathrm{D}_{2p}$ and $(\mathbb{Z}_2)^2 \times \mathbb{Z}_p \rtimes \mathbb{Z}_4 (4 \mid p-1)$, respectively. Now suppose that $g_{4p}, g_{4p+1}, g_{2p,2p} \notin G$. Then $g_p^*(s=4), g_2^*(s=2p), g_{2^2}^*(s=p) \in G$, and by Lemma 3.9, $G = \mathbb{Z}_q \rtimes \mathbb{Z}_4$.
If $m = p+1$, where p is prime, then $\mid \Omega \mid = 2p+2$ or $2p+3$. By Theorem 2.4, there exists an element $g \in G$ of prime-power order without fixed points. First suppose that $\mid \Omega \mid = 2p+2$. From Proposition 2.3, we have $o(g) = 9, 25$ or q, where q is an odd prime. Therefore $2p+2 = 18, 25$ or sq for some $s > 0$, respectively. Since p is prime, $2p+2 \neq 18, 25$. Hence $2p+2 = sq$ and $\text{move}(g) = s\dfrac{q-1}{2}$. As $s > 0$, $\text{move}(g) \neq p+1$. So $s\dfrac{q-1}{2} = p-1$. This implies that $s = 4$. Hence $p = 5, q = 3$ and G is one of the groups listed in Table 3, or from the previous case, we have $G = \mathbb{Z}_{4q}$, $\mathbb{Z}_q \rtimes \mathbb{Z}_4$ or D_{2n}, where $n = 2q$. Suppose now that $\mid \Omega \mid = 2p+3$. By the same argument there exists an element $g \in G$ of prime-power order without fixed points. So $\mathrm{o}(g) = 9, 25$ or q and $2p+3 = 18, 25$ or sq for some $s > 0$. Clearly, $2p+3 \neq 18$. if $2p+3 = 25$, then $m = 12$ and the argument given above for $m = 12$ implies that $G = \mathbb{Z}_{25}$, D_{50} or one of the groups in Table 3. If $2p+3 = sq$, where q is an odd prime, then $\text{move}(g) = s\dfrac{q-1}{2}$. If $\text{move}(g) = p+1$, then $s = 1$ and $\mathrm{o}(g) = 2p+3$ is prime. However, by **case 1** (1) every non-identity element has the same movement equal to $p+1$. Therefore $\text{move}(g) = s\dfrac{q-1}{2} = p-1$. This implies that $s = 5$. Since $4 \mid 2p+2$, we can assume $q - 1 = 2^a\ (a \geq 2)$ and G has an element g as a product of k disjoint cycles of length $q-1$. So $k = \dfrac{5q-1}{q-1} = 5 + \dfrac{4}{q-1}$. This implies that $q - 1 \mid 4$. Therefore $q = 5$ and G is a transitive group on 25 points, which is classified in the case $m = 12$.
Let $m = 2 + (s-1)2^{a-1}$ and $g_{2^a,4}^* \in G \quad (s > 1, a > 2)$. Since G is not a 2−group, there is a prime number p and an element $h \in G$ such that $p | o(h)$. By Proposition 2.3, h can be one of the elements $g_{2p,2p}, g_{2,2p}$ or g_p^*. Since we have already checked the assumption that $\text{move}(h) = m$, we only deal with $\text{move}(h) = m-2$. If $h = g_{2p,2p}, g_{2,2p}$, then $m - \text{move}(h^2) = 4$, which is a contradiction. Thus let h be the product of A disjoint cycles of length p. So $\text{move}(h) = A\frac{p-1}{2} = m-2 = (s-1)2^{a-1}$. By Theorem 2.4, we have $Ap \leq |\Omega| = 4 + (s-1)2^a$. If $Ap = |\Omega| = 4 + (s-1)2^a$, then

$$Ap = |\Omega| = 4 + (s-1)2^a = 4 + A(p-1) \implies A = 4 \Longrightarrow m = 2p.$$

If $Ap < |\Omega| = 4 + (s-1)2^a$, then $A < 4$.
Let $A = 1$, then $|\Omega| = p+3, m = \frac{p+3}{2}$ and $G = \langle g_p, g_{2^a,4}^* \rangle$, where $2^a | p-1$. Now let $A = 2$. Then $|\Omega| = 2p+2, m = p+1$, which has already been discussed. If $A = 3$, then $|\Omega| = 3p+1, m = \frac{3p+1}{2}$ and $G = \langle g_p^*(s=3), g_{2^a,4}^* \rangle$, where $2^a | p-1,\ \ 3 | s-1$.
Let $m = 2 + (s-2)2^{a-1}$ and $g_{2^a,2,2}^* \in G\ \ (s > 2, a > 1)$. A similar argument to the above paragraph implies that $|\Omega| = 4p, m = 2p$, or $|\Omega| < 4p$. The former case has been checked. In the latter case, we have $Ap < |\Omega| = 4 + (s-2)2^a = 4 + A(p-1)$. therefore $A < 4$. If $A = 1$, then $|\Omega| = p+3, m = \frac{p+3}{2}$ and $G = \langle g_p, g_{2^a,2,2}^* \rangle$, where $2^a | p-1$. If $A = 2$, then $|\Omega| = 2p+2, m = p+1,\ G = \langle g_{p,p}, g_{2^a,2,2}^* \rangle$ and $g_{2p} \notin G$, Finally, if $A = 3$, then

$|\Omega| = 3p+1, m = \frac{3p+1}{2}$, $G = \langle g_p^*(s=3), g_{2^a,2,2}^* \rangle$ where $2^a | p-1, 3 | s-2$ and $g_{3p} \notin G$. This completes the proof of Theorem 1.2.

References

[1] M. Alaeiyan and B. Askari, *Transitive permutation groups with elements of movement m or m-1*, Math. Reports **14** (64), 4 , 317-324, 2012.

[2] M. Alaeiyan and M. Rezaei, *Intransitive permutation groups with bounded movement having maximum degree*, Math. Rep. **13** (63), 109-115, 2011.

[3] M. Alaeiyan and H. Tavallaee, *Permutation groups with the same movement*, Carpathian J. Math., 147-156, 2009.

[4] T. Dokchitser, *Transitive groups of degree up to 31*, accessed on 27 july 2022. URL: GroupNames.org.https://people.maths.bris.ac.uk/ matyd/GroupNames/T31.htmltml.

[5] B. Fein, W. M. Kantor, and M. Schacher, *Relative brauer groups ii*, J. reine angew. Math **328**, 39-57, 1981.

[6] G. Group, *Gap-groups, algorithms, and programming*, version 4.11. 1, 2021. URL: https://www. gap-system. org.

[7] A. Hassani, M. Alaeiyan(Khayaty), E. Khukhro, and C. E. Praeger, *Transitive permutation groups with bounded movement having maximal degree*, J. Algebra **214** (1), 317-337, 1999.

[8] A. Mann and C. E. Praeger, *Transitive permutation groups of minimal movement*, J. Algebra **181**(3), 903-911, 1996.

[9] C. E. Praeger, *On permutation groups with bounded movement*, J. Algebra **144** (2), 436-442, 1991.

[10] J. J. Rotman, *An introduction to the theory of groups*, Number 3rd ed. Allyn and Bacon, Boston, 1984.

[11] T. Tsuzuku, *Finite groups and finite geometries*, volume 78, Cambridge University Press, 1982.