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## ON FULLY STABLE ACTS

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**Abstract.** The purpose of this paper is to introduce and investigate the fully stable acts as a concept generalizing fully stable modules but is stronger than that of duo acts. In this study, we consider some properties and characterizations of the class of fully stable acts, and the relations between this class and other well studied classes of acts, like quasi-injective acts and acts satisfying Baer's criterion.

**Keywords:** fully stable, Baer's criterion, quasi-injective, right S-act.

**Mathematics Subject Classification:** 20M30.

## 1. Preliminaries

Let  $S$  be a monoid. A right  $S$ -act  $M_S$  is a nonempty set  $M$  together with a map (written multiplicatively) from  $M \times S$  into  $M$  satisfying  $m1 = m$  and  $m(st) = (ms)t$ , for all  $m \in M$  and  $s, t \in S$ .

A nonempty subset  $N$  of an  $S$ -act  $M_S$  is  $S$ -subact if  $ns \in N$  for all  $s \in S$  and  $n \in N$ . We say that  $M_S$  is a cyclic  $S$ -act if  $M_S = uS$  for some  $u \in M_S$ .

An element  $z \in M_S$  is called a *fixed element* of  $M_S$  if  $zs = z$  for all  $s \in S$ . The set of all fixed elements of  $M_S$  will be denoted by  $\mathcal{F}(M)$ .

If  $M_S$  has a unique fixed element  $z$ , then  $z$  is called *zero element* of  $M_S$ . We will denote the zero element of  $M_S$  by  $\mathcal{O}$ . Every  $S$ -act  $M_S$  can be extended to an  $S$ -act with fixed element  $z$  by taking the disjoint union:  $M_S \dot{\cup} \{z\}_S$ .

A nonempty subset  $K \subseteq S$  is called *left ideal* of a monoid  $S$  if  $SK \subseteq K$ ; a *right ideal* of  $S$  if  $KS \subseteq K$ ; an *ideal* of  $S$  if  $KS \subseteq K$  and  $SK \subseteq K$ .

Recall that, for two  $S$ -acts  $A_S, B_S$  a mapping  $\theta : A_S \rightarrow B_S$  is called a *homomorphism of  $S$ -acts* or just an  *$S$ -homomorphism* if  $\theta(as) = \theta(a)s$  for all  $a \in A_S, s \in S$ . The set of all  $S$ -homomorphisms from  $A_S$  into  $B_S$  will be denoted by  $\text{Hom}(A_S, B_S)$  or sometimes by  $\text{Hom}_S(A, B)$ .

Note that if  $\theta : A_S \rightarrow B_S$  is an  $S$ -homomorphism then  $\text{Im } \theta = \theta(A_S)$  is a subact of  $B_S$ , and the  $S$ -homomorphism  $f : M_S \rightarrow M_S$  is called an endomorphism of  $M_S$ .

The set  $\text{Hom}_S(M, M)$  which forms a monoid under composition of mappings is denoted by  $\text{End}_S(M)$  and is called the endomorphism monoid of  $M_S$ .

An equivalence relation  $\rho$  on an  $S$ -act  $M_S$  is called an  $S$ -act congruence or a congruence on  $M_S$ , if  $(m, n) \in \rho$  implies  $(ms, ns) \in \rho$  for  $m, n \in M_S$ ,  $s \in S$ . If  $S$  is a monoid then any right (semigroup) congruence  $\rho$  on  $S$  is an act congruence on  $S_S$ . Also, for an  $S$ -act  $M_S$ ,  $H \subset S$ ,  $K \subset M \times M$ ,  $T \subset M$ ,  $J \subset S \times S$ .

$$\begin{aligned}\mathcal{L}_M(H) &= \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in H\}; \\ \mathcal{R}_S(K) &= \{s \in S \mid as = bs \text{ for all } (a, b) \in K\}; \\ \mathcal{R}_S(T) &= \{(a, b) \in S \times S \mid ma = mb \text{ for all } m \in T\}; \\ \mathcal{L}_M(J) &= \{m \in M \mid ma = mb \text{ for all } (a, b) \in J\}.\end{aligned}$$

The above is a kind of *annihilator* in  $S$ -act. Where  $\mathcal{L}_M(H)$  (resp.  $\mathcal{L}_M(J)$ ) are called the left annihilator of  $H$  (resp.  $J$ ) and  $\mathcal{R}_S(K)$  (resp.  $\mathcal{R}_S(T)$ ) are called the right annihilator of  $K$  (resp.  $T$ ).

Clearly,

$$\mathcal{R}_S(M) = \{(a, b) \in S \times S \mid ma = mb, \text{ for all } m \in M\}$$

is a right semigroup congruence on  $S$  and  $\mathcal{R}_S(K)$  is a right ideal of  $S$ . If  $S$  is commutative then the set

$$\mathcal{L}_M(S) = \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in S\}$$

is a congruence on  $M_S$  and, if  $\mathcal{L}_M(J) \neq \emptyset$ , then it is a subact of  $M_S$ .

Recall that for a family  $\{A_i\}$ ,  $i \in I$ , of right  $S$ -acts their Cartesian product  $\prod_{i \in I} A_i$  with the  $S$ -action (multiplication) defined by  $(a_i)s = (a_i s)$  is the *product* of a family of  $\{A_i\}$ ,  $i \in I$  of a right  $S$ -act.

The *coproduct* of a family of  $\{A_i\}$ ,  $i \in I$  of a right  $S$ -act is their disjoint union

$$\coprod_{i \in I} A_i = \left( \bigcup_{i \in I} A_i \times \{i\} \right)$$

with the action of  $S$  defined by  $(a, i)s = (as, i)$  for  $a \in A_i$  and  $s \in S$ .

For the family  $\{A_i : i \in I\}$  of  $S$ -acts with a unique fixed element (zero element  $\mathcal{O}$ ), the *direct sum*  $\bigoplus_{i \in I} A_i$  is defined as the subact of the product  $\prod_{i \in I} A_i$  consisting of all  $(a_i)_{i \in I}$  such that  $a_i = \mathcal{O}$  for all  $i \in I$  except a finite number. We use  $\bigoplus_{i \in I} A_i$  only when the  $S$ -acts  $A_i$  have unique fixed elements.

An  $S$ -act  $M_S$  is called *injective* if for each  $S$ -monomorphism  $g : A_S \rightarrow B_S$  (where  $A_S, B_S$  are any two  $S$ -acts) and each  $S$ -homomorphism  $f : A_S \rightarrow M_S$ , there exists an  $S$ -homomorphism  $h : B_S \rightarrow M_S$  such that  $hg = f$ .

A subact  $B_S$  is *essential* in an  $S$ -act  $M_S$  if for any  $S$ -act  $A_S$  and any  $S$ -homomorphism  $f : M_S \rightarrow A_S$  whose restriction to  $B$  is one-to-one, the map  $f$  is itself one-to-one. In such a case, we say that  $M_S$  is an essential extension of  $B_S$ . The minimal injective extension of  $M_S$  is called the injective hull of  $M_S$  and is denoted by  $E(M)$ . Note that  $E(M)$  is the injective hull of  $M_S$  if and only if  $M_S$  is essential in  $E(M)$  and  $E(M)$  is injective [3].

The *Jacobson radical* of an  $S$ -act  $M_S$  (denoted  $\mathcal{J}(M_S)$ ) is defined by:

$$\mathcal{J}(M_S) = \{m \in M_S \mid \lambda_m \text{ is one-to-one only on one element right ideals of } S\},$$

where the mappings  $\lambda_m : S_S \rightarrow M_S$  are given by  $s \mapsto ms$  for each  $s \in S$ .

For an  $S$ -act  $M_S$  with zero element  $\mathcal{O}$ , the Jacobson radical  $\mathcal{J}(M_S)$  is a subact of  $M_S$  [4].

## 2. Fully stable acts

In 1990, M.S. Abbas introduced a class of modules is called a fully stable as follows, a submodule  $N$  of an  $R$ -module  $M_R$  is called stable if  $f(N) \subseteq N$  for each  $R$ -homomorphism  $f : N \rightarrow M$ ,  $M$  is called fully stable module in case each submodule of  $M$  is stable [1].

In this section, we introduced the fully stable concept as a class of acts, and give several characterizations of fully stable acts. Also we consider the relations between this class and acts satisfying Baer's criterion.

**Definition 2.1.** Let  $M_S$  be an  $S$ -act. A subact  $N_S$  of  $M_S$  is called stable, if  $f(N) \subseteq N$  for each  $S$ -homomorphism  $f : N \rightarrow M$ . The act  $M$  is called fully stable in case each subact of  $M$  is stable. A monoid  $S$  is fully stable if it is a fully stable  $S$ -act.

We have directly from the definition that every fully stable act is duo act, where by a duo  $S$ -act  $M_S$  we mean an  $S$ -act in which every subact  $N_S$  is fully invariant (i.e.  $f(N) \subseteq N$  for any  $S$ -homomorphism  $f : M \rightarrow M$  [2]).

However, the converse need not to be true in general; for example, it is easy to see that the act  $\mathbb{Z}_{(\mathbb{Z},.)}$  of all integers is duo but not fully stable. For, if we define  $\alpha : 2\mathbb{Z} \rightarrow \mathbb{Z}$  by  $2n \mapsto 3n$ , then, clearly,  $\alpha$  is a  $\mathbb{Z}$ -homomorphism for which  $\alpha(2\mathbb{Z}) \not\subseteq 2\mathbb{Z}$  since  $\alpha(2.1) = 3.1 = 3 \notin 2\mathbb{Z}$ .

### Remarks 2.2.

1. Every subact of a fully stable act is fully stable.
2. The direct sum (hence, product) of fully stable acts need not be fully stable. For instance, let  $M_S$  be a fully stable  $S$ -act with a unique fixed element (zero element  $\mathcal{O}$ ). The map  $f : M \oplus \{\mathcal{O}\} \rightarrow M \oplus M$  defined by  $f((m, \mathcal{O})) = (\mathcal{O}, m)$  is an  $S$ -homomorphism. Hence from the definition of direct sum, there is an element  $\mathcal{O} \neq a \in M$  with  $f((a, \mathcal{O})) = (\mathcal{O}, a) \notin M \oplus \{\mathcal{O}\}$ . Thus  $f(M \oplus \{\mathcal{O}\}) \not\subseteq M \oplus \{\mathcal{O}\}$ .

3. The coproduct of any family of fully stable acts need not be fully stable. For example, let  $M_S$  be a fully stable  $S$ -act,  $N \times \{1\}$  be a subact of  $M \amalg M = M \times \{1\} \dot{\cup} M \times \{2\}$ . Define  $\theta : N \times \{1\} \rightarrow M \amalg M$  by  $\theta((n, 1)) = (n, 2)$ . Clearly,  $\theta$  is an  $S$ -homomorphism but  $\theta(N \times \{1\}) \not\subseteq N \times \{1\}$ , since for any  $n \in N$   $(n, 2) \notin N \times \{1\}$ .

In the following corollary, it is seen that to determine whether an act is fully stable it suffices to consider stability of a very restricted class of subacts.

**Corollary 2.3.** *An  $S$ -act  $M_S$  is fully stable if and only if every cyclic subact is stable.*

In the following proposition we give another characterization of fully stable acts which will be used later, when a monoid  $S$  is commutative.

**Proposition 2.4.** *An  $S$ -act  $M_S$  is fully stable if and only if for each  $x, y$  in  $M$ ,  $y \notin xS$  implies  $\mathcal{R}_S(x) \not\subseteq \mathcal{R}_S(y)$ .*

**Proof.** Suppose that  $M$  is fully stable and that there exist two elements  $x, y \in M$  with  $y \notin xS$  and  $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$ , define  $f : xS \rightarrow M$  by  $f(xr) = yr$  for  $r \in S$ . If  $xr_1 = xr_2$  where  $r_1, r_2 \in S$ , then  $(r_1, r_2) \in \mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$ . This implies that  $yr_1 = yr_2$ , hence  $f(xr_1) = f(xr_2)$ , and  $f$  is well-defined. Clearly,  $f$  is an  $S$ -homomorphism. Since  $M$  is fully stable, we have  $f(xS) \subseteq xS$  and  $y = f(x) \in xS$  which is a contradiction.

Conversely, assume that there exists a cyclic subact  $xS$  of  $M$  and an  $S$ -homomorphism  $\theta : xS \rightarrow M$  such that  $\theta(xS) \not\subseteq xS$ . Then, there exists an element  $y \in xS$  such that  $\theta(y) \notin xS$ . Let  $(s, t) \in \mathcal{R}_S(x)$ , hence  $xs = xt$ . So

$$\theta(y)s = \theta(ys) = \theta(xrs) = \theta(xsr) = \theta(xtr) = \theta(xrt) = \theta(yt) = \theta(y)t.$$

Therefore,  $(s, t) \in \mathcal{R}_S(y)$  and  $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$ , which is a contradiction. ■

It is well-known that the Jacobson radical  $\mathcal{J}(M)$  of an  $S$ -act is a fully invariant subact [3].

The following proposition gives a kind of subact which is always stable in any act.

**Proposition 2.5.** *The Jacobson radical of any act is a stable subact.*

**Proof.** Let  $M_S$  be an  $S$ -act and  $f : \mathcal{J}(M) \rightarrow M$  an  $S$ -homomorphism. If  $A$  is a right ideal of  $S$  with more than one element i.e.  $|A| \geq 2$ , then there exist  $a_1 \neq a_2 \in A$  such that  $ma_1 = ma_2$ . Hence

$$f(m)a_1 = f(ma_1) = f(ma_2) = f(m)a_2, \text{ for } m \in \mathcal{J}(M).$$

So  $\lambda_{f(m)}$  is not one-to-one on  $A$ . Thus  $f(m) \in \mathcal{J}(M)$ . ■

**Definition 2.6.** Let  $N_S$  be a subact of some act  $M_S$ . We say that  $N_S$  satisfies Baer criterion, if for every  $S$ -homomorphism  $f : N_S \rightarrow M_S$ , there exists an element  $s \in S$  such that  $f(n) = ns$  for each  $n \in N_S$ . An  $S$ -act  $M_S$  is said to satisfy Baer criterion if every subact of  $M_S$  satisfies Baer criterion.

**Proposition 2.7.** *If  $M_S$  is a fully stable  $S$ -act, then  $M_S$  satisfies Baer criterion for cyclic subacts (where  $S$  is a commutative monoid).*

**Proof.** Let  $xS$  be a cyclic subact of  $M_S$  and  $f : xS \rightarrow M$  an  $S$ -homomorphism. Since  $xS$  is stable, we have  $f(xS) \subseteq xS$  and hence  $f(x) \in xS$  i.e. there is  $t \in S$  such that  $f(x) = xt$ . Let  $w \in xS$ , hence  $w = xr$  for some  $r \in S$  and hence  $f(w) \in xS$ . So

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

Hence there is  $t \in S$  such that  $f(w) = wt$  for every  $w \in xS$ . Thus Baer criterion holds for cyclic subacts. ■

In the following proposition and its corollary, we obtain another characterization of fully stable acts. We assume the monoid  $S$  is commutative.

**Proposition 2.8.** *For an  $S$ -act  $M_S$ , Baer criterion holds for cyclic subacts if and only if  $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$  for all  $x \in M$ .*

**Proof.** Assume that the Baer criterion holds for cyclic subacts of  $M_S$ . Let  $y \in \mathcal{L}_M(\mathcal{R}_S(x))$  and define  $\theta : xS \rightarrow M$  by  $\theta(xr) = yr$  for each  $r \in S$ . If  $xr_1 = xr_2$  where  $r_1, r_2 \in S$ , then  $(r_1, r_2) \in \mathcal{R}_S(x)$ , hence  $yr_1 = yr_2$  (since  $y \in \mathcal{L}_M \mathcal{R}_S(x)$ ). Thus  $\theta$  is well-defined. It is clear that  $\theta$  is an  $S$ -homomorphism. By the assumption, there exists an element  $t \in S$  such that  $\theta(w) = wt$  for each  $w \in xS$ .

In particular,

$$y = \theta(x) = xt \in xS.$$

This implies that  $\mathcal{L}_M(\mathcal{R}_S(x)) \subseteq xS$ ; since the inclusion  $xS \subseteq \mathcal{L}_M(\mathcal{R}_S(x))$  is always true. Hence

$$\mathcal{L}_M(\mathcal{R}_S(x)) = xS.$$

Conversely, assume that  $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$  for each  $x \in M$ . Then, for each  $S$ -homomorphism  $f : xS \rightarrow M$  and  $(s, t) \in \mathcal{R}_S(x)$ , we have

$$xs = xt \text{ and } f(x)s = f(xs) = f(xt) = f(x)t.$$

Thus  $f(x) \in \mathcal{L}_M(\mathcal{R}_S(x)) = xS$ . Therefore,  $f(x) = xt$  for some  $t \in S$ . Now, for each  $w \in xS$  there exists  $r \in S$  such that  $w = xr$ , hence

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

So there exists  $t \in S$  such that  $f(w) = wt$  for each  $w \in xS$ . ■

As we have mentioned earlier, any fully stable  $S$ -act satisfies Baer criterion for cyclic subacts, thus we have the following corollary.

**Corollary 2.9.** *An  $S$ -act  $M_S$  is fully stable if and only if  $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$  for each  $x \in M$ .*

The results of this section can be summarized together with those of section one, in the following theorem.

**Theorem 2.10.** *The following statements are equivalent for an  $S$ -act  $M_S$ .*

1.  $M_S$  is a fully stable act.
2. Every cyclic subact of  $M_S$  is stable.
3. For each  $x, y$  in  $M_S$ ,  $y \notin xS$  implies  $\mathcal{R}_S(x) \not\subseteq \mathcal{R}_S(y)$ .
4.  $M_S$  satisfies Baer criterion for cyclic subacts.
5. For each  $x$  in  $M_S$ ,  $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ .

Another characterization of fully stable acts is given here.

**Remark 2.11.** An  $S$ -act  $M_S$  is fully stable if and only if for each  $S$ -act  $A_S$  and for any two homomorphisms  $f, g : A \rightarrow M$ , with  $g$  injective (one-to-one mapping), we have  $\text{Im } f \subseteq \text{Im } g$ .

**Proof.** ( $\Rightarrow$ ) Let  $A_S$  be an  $S$ -act and  $f, g : A \rightarrow M$   $S$ -homomorphisms. By injectivity of  $g$ , there exists an  $S$ -homomorphism  $h : g(A) \rightarrow A$  such that  $h \circ g = id_A$ . Since  $g(A)$  is a subact of  $M$ , we have  $g(A)$  is stable. Hence  $f \circ h(g(A)) \subseteq g(A)$ . So  $f(h \circ g(A)) \subseteq g(A)$  and  $f(A) \subseteq g(A)$ . ( $\Leftarrow$ ) Let  $N_S$  be a subact of  $M_S$  and  $f : N \rightarrow M$  an  $S$ -homomorphism. Since the inclusion  $i : N \rightarrow M$  is an injective homomorphism, we get  $f(N) \subseteq i(N) = N$ . Thus,  $M_S$  is fully stable. ■

### 3. Fully stable and quasi-injective acts

Recall that an  $S$ -act  $A_S$  is called quasi-injective [3] if for each subact  $B_S$  of  $A_S$  and any  $S$ -homomorphism  $f : B_S \rightarrow A_S$  there exists an  $S$ -homomorphism  $g : A_S \rightarrow A_S$  extending  $f$ . We will discuss the relation between quasi-injective and fully stable acts under the assumption that the monoid  $S$  is commutative. First, we recall some concepts needed. Given some concrete category  $C$ , an object  $K \in C$  is called a *cofree* object in  $C$  if there exists  $I \in \text{Set}$  and a mapping  $\psi : [K] \rightarrow I$  such that the following universal property is valid: for every  $X \in C$  and every mapping  $\xi : [X] \rightarrow I$  there exists exactly one  $\xi^* \in \text{Mor}_C(X, K)$  such that the following diagram in  $\text{Set}$  is commutative:

$$\begin{array}{ccc}
 I & \xleftarrow{\xi} & [X] \\
 \uparrow \psi & & \swarrow [\xi^*] \\
 [K] & & 
 \end{array}$$

We write  $Cof(I)$  for  $K$  and say that  $K$  is  $I$ -cofree. The set  $I$  is called a *cobasis* for  $K$ .

For the cofree concept in  $S$ -Act, we have the following proposition. But, first, recall that  $I^S = \text{Hom}({}_S S_{\{1\}}, {}_{\{1\}} I_{\{1\}})$  is a right  $S$ -act and  $fs$  for  $f \in I^S$ ,  $s \in S$  is defined by  $(fs)(t) = f(st)$  for every  $t \in S$ ,  $I \neq \emptyset$ , see [3, Remark 1.7.20].

**Proposition 3.1.** [2, p.151] *Let  $I \neq \emptyset$ . The  $S$ -act  $I^S$  with  $\psi(f) = f(1)$  for all  $f \in I^S$  is an  $|I|$ -cofree object in  $\text{Act-}S$ .*

The next proposition shows that cofree of a fully stable act is itself a fully stable act.

**Proposition 3.2.** *If the  $S$ -act  $M_S$  is a fully stable act, then  $(M^S)_S$  is fully stable (where  $S$  is a commutative monoid, i.e., the left  $S$ -act  $S$  is right).*

**Proof.** Let  $f, g \in M^S$  such that  $\mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$ , where

$$\mathcal{R}_S(g) = \{(s, t) \in S \times S \mid gs = gt\} \text{ and } \mathcal{R}_S(f) = \{(s, t) \in S \times S \mid fs = ft\}.$$

Since  $M$  is a cobasis of  $M^S$ , there exists an  $S$ -homomorphism  $\psi : M^S \rightarrow M$  such that  $\psi(f) = f(1)$ , for each  $f \in M^S$ . Hence  $f(1), g(1) \in M$  and

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

Since, if  $(s_1, s_2) \in \mathcal{R}_S(g(1))$ , then  $g(1)s_1 = g(1)s_2$ , hence  $g(s_1) = g(s_2)$  and hence  $g(s_1)(1) = g(s_2)(1)$ . Now, for each  $t \in S$ , we have that

$$g(s_1)(t) = g(s_1)(1)t = g(s_2)(1)t = g(s_2)(t)$$

by commutativity of  $S$ , hence  $(s_1, s_2) \in \mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$ , so that  $(s_1, s_2) \in \mathcal{R}_S(f(1))$ . Thus,

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

By full stability of  $M$ , we have

$$f(1)S \subseteq g(1)S.$$

Therefore,  $f \in gS$  and, by Proposition 2.4, we have that  $M^S$  is fully stable  $S$ -act. ■

Now, we ask the following question. Is there a relation between fully stable acts and quasi-injective acts? The following theorem answers this question.

**Theorem 3.3.** *Every fully stable act is quasi-injective.*

**Proof.** Let  $M_S$  be a fully stable act. Hence, for any subact  $N_S$  of  $M_S$  and  $S$ -homomorphism  $\alpha : N \rightarrow M$ , we have that  $\alpha(N) \subseteq N$ , that is  $\alpha : N \rightarrow N$ .

By the injectivity of  $E(M)$ , the map  $\alpha$  extends to an  $S$ -homomorphism  $\beta : E(M) \rightarrow E(M)$ . But  $(M^S)_S$  is a cofree fully stable  $S$ -act, hence  $(M^S)_S$  is injective fully stable act see Theorem 3.1.5 in [3], but  $E(M)$  is a minimal injective extension of  $M_S$ , hence  $E(M)$  is a subact of  $(M^S)_S$  and since every subact of fully stable is fully stable [Remark 2.2.1], hence  $E(M)$  is a fully stable act and then  $\hat{\beta} : M \rightarrow M$  is an extension of  $\alpha$  where  $\hat{\beta} = \beta|_M$ . Therefore,  $M$  a quasi-injective. ■

**Corollary 3.4.** *The injective hull of fully stable act is fully stable.*

The converse of Theorem 3.3 is not true in general as in the following example.

**Example 3.5.** Let  $S = \{0, 1\}$ . Consider the  $S$ -act  $A = \{\mathcal{O}, a, b, c\}$  with multiplication  $0 = b0 = c0 = \mathcal{O}$ . The act  $A_S$  is injective, so it must be quasi-injective. But it is not fully stable, because  $aS = \{\mathcal{O}, a\} \neq \mathcal{L}_A(\mathcal{R}_S(a)) = A$ .

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