بسم الله الرحمن الرحيم

Shahid Chamran University of Ahvaz, Ahvaz, Iran

Faculty of Mathematical Sciences and Computer

Thesis:

Submitted in Fulfillment of Requirements for award of PHD Degree in Science Mathematics

Thesis Title:

Investigating on Stability in Optimal control systems

By:

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Abstract

In this thesis, the conditions of stability, controllability and observability of linear control systems have been investigated. Stability is an important concept in control and optimal control systems. There are different methods to check the stability of a system. We have presented a new method to check the stability of a linear system using the sign of eigenvalues using two methods, "Routh" and "Lyapunov".

Using "MATLAB" software, algorithms for checking the controllability, observability and stability of a linear system are presented in a way that makes it easy for users to use.

Shahid Chamran University of Ahvaz, Ahvaz, Iran

Faculty of Mathematical Sciences and Computer

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Dissertation of Ali Farhan Hashoosh, Ph.D. student of Mathematics - Optimal Control

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Thesis Title:

Investigating on Stability in Optimal control systems

To obtain a degree: PHD. in Mathematics was evaluated by the jury and approved with a grade of (   )

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بسمه تعالی

گزارش نتیجه (ارزشیابي) دفاع از پایان نامه دکتري (ph.D)

ریاست محترم دانشکده علوم ریاضي و کامپیوتر

احتراماً ، بدینوسیله به اطلاع مي رساند آقای علی فرحان حاشوش شماره دانشجويي ۹۹۹۹۹۴۰۰۲مراحل آموزشي و پژوهشي دوره دکتري رشته ریاضي گرایش کاربردی را طبق آئین نامه مربوطه با موفقیت به اتمام رسانیده و نمره قبولي زبان را نیز کسب کرده است ، همچنین نامبرده داراي مقاله علمي در مجلات معتبر مي باشد. با توجه به **اینکه مدیریت محترم تحصیلات تکمیلي دانشگاه طي نامه شماره۱۴۰۲/۴۷/۳/۲۶۰۸۲۲ مورخ ۱۴۰۲/۷/۲۴** با برگزاري جلسه دفاع از رساله ایشان موافقت نموده اند، بنا براین در تاریخ ۱۴۰۲/۸/۷ جلسه دفاع از رساله ي ایشان زیر نظر هیئت داوران با ریاست استاد راهنما به عمل آمد. در این جلسه دانشجو گزارشي از کارهاي تحقیقاتي و رساله خود ارائه نموده و از آن دفاع کرد. هیئت داوران در پایان پس از شور و بررسي در خصوص اصالت و صحت رساله ، امتیاز آن را به شرح زیر مشخص نمود:

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	دکترمحمد رضا زادکرمی	استاد	ناظر تحصیلات تکمیلي	دانشگاه شهید چمران اهواز	

خواهشمند است دستور فرمایند مراتب فوق پس از تأئید به اطلاع مدیریت محترم تحصیلات تکمیلي دانشگاه رسانیده شود.
مدیرگروه ریاضي نام و نام خانوادگي مهدی جلالوند امضا و تاریخ

۰۲٫۸٫۸
ثبت شد

گواهی صحت و اصالت پایان‌نامه

عنوان رساله: بررسی پایداری در سیستم های کنترل بهینه

اینجانب **علی فرحان حاشوش** دانشجوی **دکتری گروه ریاضی و کامپیوتر** دانشگاه شهید چمران به شماره دانشجویی 999994002 تحت راهنمايي **دکتر هادی بصیرزاده** گواهي مي‌دهم که:

1- تحقیقات ارائه شده در این پایان‌نامه حاصل مطالعات علمی و عملی شخص اینجانب بوده و صحت و اصالت تمام مطالب مندرج در آن را تأیید می‌کنم.

2- در صورت استفاده از آثار دیگران، مشخصات کامل آن‌ها را در منابع ذکر نموده‌ام.

3- تاکنون مطالب درج شده در این پایان‌نامه، توسط اینجانب یا شخص دیگری به منظور اخذ هر نوع مدرک یا امتیازی به هیچ مرجعی تسلیم نشده و بعد از این نیز نخواهد شد.

4- در تدوین متن پایان نامه، شیوه‌نامه مصوب دانشکده را رعایت نموده‌ام.

5- کلیه حقوق معنوی این اثر به دانشگاه شهید چمران تعلق داشته و مقالات مستخرج از آن، ذیل نام دانشگاه شهید چمران اهواز (Shahid Chamran University of Ahvaz) به چاپ خواهد رسید.

6- حقوق معنوی تمامی افرادی که در این پایان‌نامه تأثیرگذار بوده‌اند (اساتید راهنما) در مقالات مستخرج از آن رعایت خواهد شد.

7- در صورت استفاده از موجودات زنده یا بافت‌های آن‌ها، کلیه ضوابط و اصول اخلاقی مندرج در منشور موازین و اصول اخلاق پژوهش وزارت علوم، تحقیقات و فناوری رعایت شده است.

8- در کلیه‌ی مراحل انجام این پایان نامه، در مواردی که به حوزه ی اطلاعات شخصی افراد دسترسی یافته و یا استفاده شده است، اصل رازداری، ضوابط و اصول اخلاقی انسانی رعایت شده است.

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علی فرحان حاشوش

(بسم الله الرحمن الرحيم)

﴿وَلَمَّا بَلَغَ أَشُدَّهُ وَاسْتَوَى آتَيْنَاهُ حُكْمًا وَعِلْمًا وَكَذَلِكَ نَجْزِي الْمُحْسِنِينَ﴾

(القصص: 14)

– صدق الله العلي العظيم –

Thanks, and appreciation

First, thanks to God, before and after, for countless blessings and virtues. I can only say

﴿رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَدْخِلْنِي بِرَحْمَتِكَ فِي عِبَادِكَ الصَّالِحِينَ﴾ (النمل: 19)

Thanks to the presidency of the University of Shahid Chamran, and in particular thanks to the of Faculty Mathematical Sciences and Computer and its generous teaching staff. I would like to express my deep gratitude and thanks to (Dr. Hadi Basirzadeh) for accepting the task of supervising this research and for his valuable guidance and comments that he provided me during the period of preparing this research. I extend my thanks and appreciation to everyone who had the credit for producing this research in this form, even with an idea or advice. Finally, I extend my thanks to my professors and members of the discussion committee for giving of their valuable time and for what they will add to this research in terms of the richness of their knowledge, the originality of their thought, and for providing me with valuable comments that will, God willing, have a beneficial impact in reaching this research to a good goal.

by

Ali Farhan hashoosh

Dedication

To those who supported my faltering steps: I dedicate this Thesis to the two symbols of giving and love: my mother and father.

To everyone who was with me and supported me throughout my scientific career.

Contents

1-Introduction

This thesis contains four chapter: In the first chapter, basic concepts for linear control systems were studied as a beginner to study the stability, controllability, and observability of these systems. Some differential equations that are frequently used in linear control systems were clarified, as well as the classification of control systems. A very important concept in building linear systems called feedback was also clarified. Two important concepts were also clarified in a new and clear way (Transfer functions and State space model).

In the second chapter of this thesis, our goal is to clarify the concepts of observability and controllability and to give different methods for calculating them. More than one method was given, including a method using an algorithm in MATLAB for both concepts.

We know the problems we face when knowing the stability of linear control systems and the difficulty of understanding the stability of some systems. Some systems are unstable and cannot be made stable, and some systems are continuously stable. In order to overcome the above difficulties, a new method (Lyapunov Routh stability method) of measuring the stability of systems was presented. Linearity in the third chapter of this thesis, the well-known extrapolation methods were explained more smoothly (Routh-Hurwitz Stability Criterion, Lyapunov Stability, Roth Lyapunov and Nyquist method).

In the fourth chapter of this thesis, two multi-population mathematical models were chosen for the first time in this thesis, the stability of a model for cancer disease and its control has been demonstrated in the other model, its stability was clarified and described in such a way that to be easy for the reader to understand.

2-Previous studies

The study of optimal control greatly attracted the attention of many mathematicians because of the necessity of strict expression form in optimal control theory. In last several decades, optimal control theory has achieved plenty of developments not only in theory [1] but also in applications such as economics, production engineering and management [2]. An optimal control problem for a given system is to choose the best decision such that an objective function is optimize.

As is well known, this problem has very important applications in practice. From 1970s many researchers began to investigate stochastic optimal control problem, such as in Merton [3] for finance. In recent decades, the study of stochastic optimal control has made considerable advances, for example, Fleming and Rishel [4], Harrison [5], Karatzas [6] and Cairns [7] studied optimal control problems of Brownian motion or stochastic differential equations and applications in finance and engineering. One of the main methods to study optimal control is based on dynamic programming. The utilization of dynamic programming in optimization over Ito's process was discussed in Dixit and Pindyck [8]. The complexity of the real world makes the events we face uncertain in various forms. Lots of human uncertainty do not behave like randomness, such as the price of a new stock, oil field reserves and bridge strength. In order to deal with these phenomena, an uncertainty theory was established in 2007 [9] and refined by Liu [10] in 2010 as a branch of axiomatic mathematics for modeling human uncertainty. Furthermore, Liu [11] introduced uncertain process and canonical process as counterparts of stochastic process and Wiener process, respectively. Then the concept of uncertain differential equation was presented in 2008 [11].

As we all know, stability of a system is a very fundamental and important problem in optimal control theory. For uncertain systems, Liu [12] introduced the concept of stability in measure in 2009, and then a sufficient condition concerning stability in measure was established by Yao et al. [13]. In 2014, Liu and Fei [14] proposed the definition of almost surely stability and also presented a sufficient condition for judging this stability. Yao et al. [15] defined stability in mean, and studied the contact between stability in mean and stability in measure, and then gave a sufficient condition about stability in mean. These three different stabilities can be employed to describe the internal characters of uncertain systems in different aspects, respectively. In 2015, Tao and Zhu [16] investigated attractivity and stability of uncertain differential systems. These results on stability and attractivity have provided solid theoretical basis for studying optimal control problem of uncertain systems. Based on uncertain differential equation, Zhu [17] investigated the excepted value model of uncertain optimal control problem in 2010. Employing Bellman's principle of optimality, he obtained an equation of optimality as a counterpart of HJB equation, and then solved an uncertain portfolio selection problem. Moreover, by the equation of optimality, Yao and Qin [18] proposed an uncertain linear quadratic control model in 2011. In 2012, Xu and Zhu [19], Kang and Zhu [20] studied uncertain bang–bang control problems for continuous-time system and multi-stage system, respectively. Sheng and Zhu [21] discussed the optimistic value model of uncertain optimal control in 2013. Li and Zhu [22] investigated uncertain linear systems and its applications in 2015. Singular systems, also known as descriptor systems, implicit systems and generalized state-space systems, are described by differential-algebraic equations (DAEs). Singular systems [23,24,] have been extensively studied during the past decades due to the fact that they are able to describe plenty of natural phenomena in physical systems such as economics, demography, microelectronic circuits and so on [25,26,27].

3-Description of the research method

We also know that there are many optimal control systems that we need to know their stability, Observability and controllability. In this dissertation, we will explain in detail and try to write MATLAB programs to find the observability and controllability of linear control systems and we will choose some models of optimal control systems and study the stability of these systems, and there are other unstable models that we will try to make stable by relying on methods of induction in optimal control (Routh stability criterion, Lyapunov's Method, Nyquist stability criterion...)
We investigate regularity and stability properties of the solution of the following optimal control problem:

$$\min g\big(x(T)\big) \qquad (0.1)$$

Subject to the linear dynamics

$$\dot{x} = A(t)x(t) + B(t)u(t) + d(t) \qquad , x(0) = x_0 \qquad (0.2)$$

$$u(t) \in U \qquad (0.3)$$

Here $x \in R^n, u \in U \subset R^r$, the time interval $[0, T]$ is fixed, $g: R^n \to R$ is smooth and convex, $A \; and \; B$ are smooth matrix functions with appropriate dimensions. The initial state x_0 is given. The control constraining set $U \subset R^r$ is a convex compact polyhedron. As usual, a dot above a symbol denoting a function of the time t means the time-derivative. Optimal control problems for linear systems have been profoundly studied since the early days of the optimal control theory but there are issues of interest that are recent research topics or are still open. In particular, this concerns the stability analysis of the optimal solution, which is burdened by the fact that the optimal control is discontinuous (bang-bang). This may be the case also for optimal control problems that are non-linear, but affine with respect the control. The "$bang - bang$" structure of the optimal control brings a challenge also for numerical approximations. We refer to the recent papers [28, 20, 1] on stability analyses and to $[26, 7, 29, 11]$ about error analyses for problems with bang-bang solutions, Therefore, the study will be limited to linear control systems.

We analyze the stability of the control problem $(0.1)\ and\ (0.3)$ through the following necessary optimality conditions: any optimal pair $(\hat{x}, \hat{u})$ together with a corresponding absolutely continuous function
$p : [0, T] \to R^n$ Satisfies the following (generalized) equations:

$$\dot{x} - A(t)x(t) - B(t)u(t) - d(t) = 0 \quad , x(0) = x_0 \tag{0.4}$$

$$\dot{p}(t) + A^T(t)p(t) = 0 \tag{0.5}$$

$$B^T(t)p(t) + N_U(u(t)) = 0 \tag{0.6}$$

$$p(T) - \nabla g(x(T)) \tag{0.7}$$

Where $N_U(u(t))$ is the normal cone to U defined as

$$N_U\big(u(t)\big) = \begin{cases} \emptyset & if\ u \notin U \\ \{l \in R^r : \langle l, v - u\rangle \le 0, \forall v \in U\} & if\ u \in U \end{cases}$$

$(Note\ that\ (6) is\ equivalent\ to\ u(t) \in \ \mathrm{Argmin}_{w \in U}\langle B^T(t)p(t), w\rangle)$

Then the following question is relevant for the stability of the solution of problem $(0.1)\ and\ (3)$: if the left-hand side of $(0.4) and (0.7)$ is replaced with a vector $y = (\xi, \pi, \rho, \nu)$, does the resulting perturbed version of $(0.4) and\ (0.7)$ still have a solution (x, p, u), and how far is it from the solution $(\hat{x}, \hat{p}, \hat{u})$of the original system $(0.4)\ and\ (0.7)$.

The answer of the first question is apparently positive, while one of the main results in gives $a\ H\ddot{}older\ estimation$ for the solution (s) (x, p, u) corresponding to disturbance y in a neighborhood of zero:

$$dist\ ((\hat{x}, \hat{p}, \hat{u}), (x, p, u)) \le c\|y\|^{1/k} \tag{0.8}$$

One of the aims of this Thesis is to correctly define the meaning of the "$neighborhood$", the norm$\|.\|$, the metric "$dist$" (and the respective spaces), and the number k for which the estimation (0.8) holds.

A related question is whether the estimation (0.8) is stable with respect to perturbations itself. It turns out that in the context of system $(0.4)\ and\ (7)$ the stability of estimation (0.8) is valid for perturbations that are small in a substantially stronger norm, $\|.\| \sim \ge \ \|.\|$, than the one in the right-hand side of (0.8). We grasp this phenomenon in general, by defining the so-called strong bi-metric $H\ddot{}older$ regularity. An inverse function theorem is proved for strongly bi-metrically regular mappings in the Lipschitz case$k = 1$.

For our particular system $(0.4)\ and\ (0.7)$ we give a sufficient condition for strong bi-metric $H\ddot{}older$ regularity, where the natural number k is the so-called controllability index of the solution $(\hat{x}, \hat{u})$ of the original problem (0.1) and (0.3). The metric "dist" in which we compare the controls $\hat{u}$ and u, in particular, is defined (in view of the bang-bang structure of $\hat{u}$) as the measure of the set where $u(t) \neq \hat{u}(t)$.

Using the proved inverse function theorem, we obtain that in the Lipschitz case $k = 1$ the strong metric bi-regularity of (0.4) and (0.7) is preserved under sufficiently "small" perturbations that can be non-linear in x.
As a byproduct we obtain the (somewhat surprising) fact that the nonlinear optimal control problem resulting from such perturbations has no" singular arcs" ($o$ptimal arcs which are not uniquely determined by the Pontryagin system). In the general case $k \geq 1$ we also provide a stability result of system $(0.4) and$ (0.7) (and the underlying problem (0.1) and (0.3)) with respect to perturbations in the matrices A and B which are small in suitable norms.
In spite of the fact that the theory of stability for stochastic hereditary systems is very popular in researches, in literature there are some simply and clear formulated problems, solutions of which have not been found so far.
three problems of such type are presented: two stability problems for stochastic differential equations with delay and one problem for stochastic difference equation with continuous time, in particular, definitions of stochastic differential and difference equations, definitions of solution stability.

4- Aims of this research

The main objective of the research is to know when the system is stability, controllability and observability, and answer the following questions:
a) Question that may arise for this research
1) Are the Controllable necessarily Observable?
2) Are the Controllable necessarily unobservable?
3) Are the Uncontrollable necessarily observable?
4) Are the Uncontrollable necessarily unobservable?
5) Are the stable systems necessarily controllable?
6) Are the stable systems necessarily observable?
7) Are the unstable systems necessarily uncontrollable?
8) Are the unstable systems necessarily unobservable?
b) Response to the question through this research.

Chapter one

Basic concepts

1-1 Introduction

A control system is a system of devices that mange, commands, directs or regulates the behavior of other devices (CD players and automobiles or in industrial robots and air plane autopilots, ...) to achieve a desired result. In other words, the definition if control system con be simplified as system that control other system to achieve the desired state. There are different types of control systems which con broadly be classified as linear control systems or nonlinear control systems, in this thesis our study will be limited on the linear control systems, in order understand the principle of superposition the principle of superposition theorem includes two the important properties and they are explained below homogeneity (a system is to be homogeneity if multiply input with some constant λ then the output will also be multiplied by the same value of constant (i.e λ))and additivity (suppose have system S and we are giving the input to system as x_1 for the first time and we are getting the output as y_1 corresponding to input x_1 on second time we are giving input x_2 and correspond to this we are getting the input y_2 now suppose this time we are giving input as a summating of the previous input $(i.e\ x_1 + x_2)$ and correspond to this input suppose we are getting the output $(y_1 + y_2)$ then we can say that system S is following the property of additivity .now we are able to define the linear control system as those types of control system which follow the principle of homogeneity and additivity[30].

A control system is a system, which provides the desired response by controlling the output. The following figure 1.1 shows the simple block diagram of a control system.

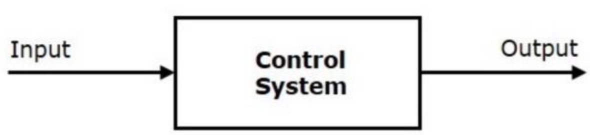


Figure 1.1

Here, the control system is represented by a single block. Since, the output is controlled by varying input, the control system got this name. We will vary this input with some mechanism. In the next section on open loop and closed loop control systems, we will study in detail about the blocks inside the control system and how to vary this input in order to get the desired response.

From examples about control system, the traffic lights control system, washing machine traffic lights control system is an example of control system. Here, a sequence of input signal is applied to this control system and the output is one of the three lights that will be on for some duration of time. During this time, the other two lights will be off. Based on the traffic study at a particular junction, the on and off times of the lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights control system operates on time basis. The washing machine control system is an example of a control system as well. It puts inside the washing machine the clothes to be washed and sets it to a certain time. When the required time is reached, the washing machine is turned off and an indication is given to the user that the work has been completed by issuing a specific sound or sometimes by the path of particular light. So, the washing machine system operates on time basis.

1-2 Differential Equation Model (some system equations)

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

We will now explain some system equations to obtain a mathematical control system some system equations:

1) electromagnetic theory it is known that, for a coil (Fig. 1.2) having an inductance L the electromotive force (e.m.f) E is proportional to the rate of change of the current I,at the instant considered, that is,

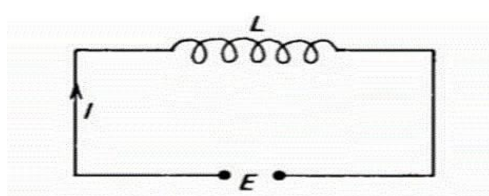


Figure 1.2

$$E = L\frac{dI}{dt} \quad or \; I = \frac{1}{L}\int E\,dt \qquad (1.1)$$

2) electrostatics theory: similarly, form electrostatics theory we know (Figure 1.3) that the voltage V and current I through a capacitance C are related at the instant t by:

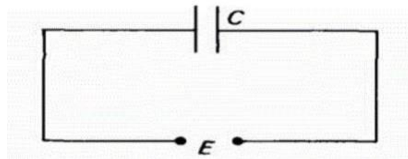


Figure 1.3

$$I = C\frac{dE}{dt} \quad or\ E = \frac{1}{C}\int I\,dt \qquad (1.2)$$

3) A dashpot device which consists of a piston sliding in an oil filled cylinder.

The motion of the piston relative to the cylinder is resisted by the oil, and this viscous drag can be assumed to be proportional to the velocity of the piston. If the applied force is $f(t)$ and the corresponding displacement is $y(t)$, then Newton's Law of motion is

$$F = \mu\frac{dy}{dt} \quad or \quad y(t) = \frac{1}{\mu}\int F\,dt \qquad (1.3)$$

Where the mass of the piston is considered negligible, and μ is viscous damping coefficient.

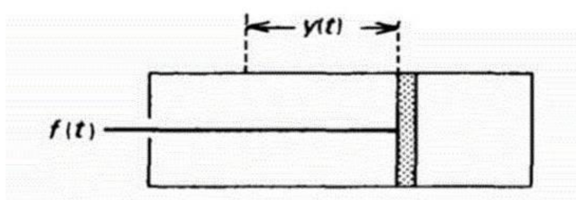


Figure 1.4

4) to analyze the control of the level of a liquid in a tank we must consider the input and output regulations, Figure 1.5 shows a tank with

q_i: *inflow rate*

q_0: *outflow rate*

h: $head\ level$

A: the cross-section area of the tank

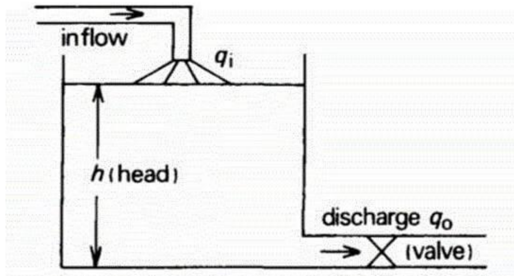


Figure 1.5

If $q = q_0 - q_i$ is the net rate of inflow into tank, over a period δ_t, and δ_h is corresponding change in the head level, then $q\delta_t = A\delta_h$ and in tle limit as $\delta_t \to 0$

$$q = A\frac{dh}{dt} \quad or \;\; h = \frac{1}{A}\int q\, dt \qquad (1.4)$$

We could produce many more examples of very diverse situations in economics, physiology, etc. in which the dynamic behavior of the system can be shown to be characterized b a differential equation similar to the above. All these equations

$$E = L\frac{dI}{dt} \qquad (1.1)$$

$$I = C\frac{dE}{dt} \qquad (1.2)$$

$$F = \mu\frac{dy}{dt} \qquad (1.3)$$

$$q = A\frac{dh}{dt} \qquad (1.4)$$

have something in common - they can all be written in the form

$$x = \alpha\frac{dy}{dt}$$

that is

$in\ (1.1)\ x = E\ ,\ \ \alpha = L\ and\ \ y = I$

$in\ (1.2)\ \ x = I\ ,\ \ \alpha = C\ and\ \ y = E$

$in\ (1.3)\ x = F,\ \ \alpha = \ \mu$

$in\ (1.4)\ x = q\ ,\ \ \alpha = \ A\ and\ \ y = h$

Equation (1.5) is interesting not only because its solution is also the solution to any one of the systems considered, but also because it shows the direct analogies which can be formulated between quite different types of components and systems. This has very important implications in mathematical modelling because solving a differential equation leads to the solution of a vast number of problems in different disciplines, all of which are modelled by the same equation [31].

For a more complicated system, consisting of two interconnected tanks, we must obtain explicated an equation for the outflow rate q_0 (see Figure 1.5) in terms of the resistance offered by the discharge valve. this in fact is an interesting problem which uses a 'linearization' technique a mathematical simplification of a type frequently used to obtain an approximate solution.

Over small ranges of the variables in volved the loss of accuracy may be very small and the simplification of calculations may be great. It is known that the flow rate through a restriction such as a discharge valve is of the form

$$q = Vp^{\frac{1}{2}}$$

Where p is the pressure across the valve and V is a coefficient dependent on the properties of liquid and the geometry of the valve.

Figure 1.6 shows the relation between the pressure and the flow rate and also indicates the assumption we make, that is, that in the neighborhood of the pressure

$$p = p_1$$

$$\frac{change\ in\ pressure}{change\ in\ flow\ rate} \cong \grave{R}$$

where$\grave{R}$ is a constant called the resistance of the valve at the point (of pressure) considered.

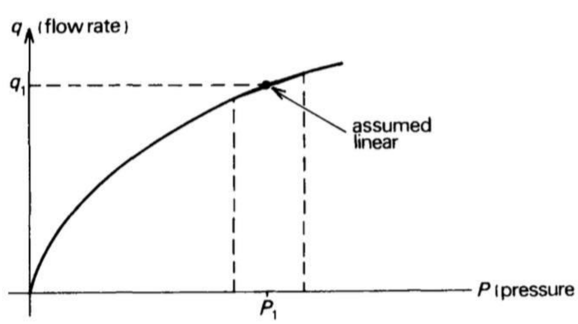


Figure 1.6

This type of assumption, which assumes that an inherently nonlinear situation can be approximated by a linear one albeit in a restricted range of the variable, is fundamental to many applications of control theory. Since, at a given point, the pressure

$$p = h\rho g$$

(p is the density of the liquid and $g = 9.8\, m/s^2$) we can write the above relation as

$$R = h/q_0 \qquad (1.6)$$

Where

$$R = \frac{\grave{R}}{\rho g}$$

A two-tank system with one inflow (q) and two discharge valves ($R_1 and\ R_2$) is shown in

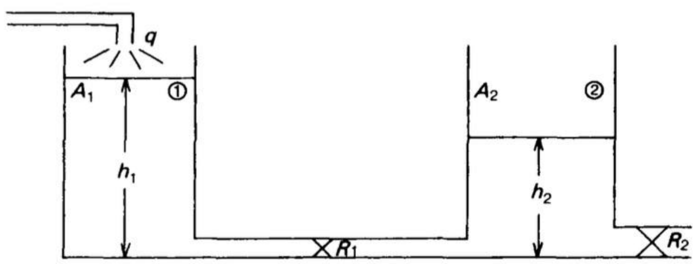


Figure 1.7

for tank 1

$$A_1 \frac{dh_1}{dt} = q - \frac{1}{R_1}(h_1 - h_2)$$

or

$$\dot{h_1} = \frac{1}{A_1} q - \frac{1}{A_1 R_1} h_1 + \frac{1}{A_1 R_1} h_2 \qquad (1.7)$$

$$where\ \dot{h_1} = \frac{dh_1}{dt}$$

for tank 2

$$A_2\frac{dh_2}{dt}=\frac{1}{R_1}(h_1-h_2)-\frac{1}{R_2}h_2$$

or

$$\dot{h_2}=\frac{1}{A_2R_1}h_1-\frac{1}{A_2}(\frac{1}{R_1}+\frac{1}{R_2})h_2 \qquad (1.8)$$

$where$

$$\dot{h_2}=\frac{dh_2}{dt}$$

$We\ can\ write\ equations\ (1.5) and\ (1.6) in\ matrix\ form\ as$

$$\begin{bmatrix}\dot{h_1}\\ \dot{h_2}\end{bmatrix}=\begin{bmatrix}-\frac{1}{A_1R_1} & \frac{1}{A_1R_1}\\ \frac{1}{A_2R_1} & -\frac{1}{A_2}(\frac{1}{R_1}+\frac{1}{R_2})\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}+\begin{bmatrix}\frac{1}{A_1}\\ 0\end{bmatrix}q$$

$that\ is,\ in\ the\ form$

$$\dot{h}=Ah+Bq \qquad (1.9)$$

$where$

$$h=\begin{bmatrix}h_1\\ h_2\end{bmatrix},\ A=\begin{bmatrix}-\frac{1}{A_1R_1} & \frac{1}{A_1R_1}\\ \frac{1}{A_2R_1} & -\frac{1}{A_2}(\frac{1}{R_1}+\frac{1}{R_2})\end{bmatrix}\ and\ B=\begin{bmatrix}\frac{1}{A_1}\\ 0\end{bmatrix}$$

We can also write the set of simultaneous first order equations ((1.7) and (1.8)) as one differential equation of the second order. On differentiating equation (1.8), we obtain

$$\ddot{h}=\frac{1}{A_2}\dot{h_1}-\frac{1}{A_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\dot{h_2} \qquad (1.10)$$

Substituting $\dot{h_1}$ from equation (1.7) and h_1, from equation (1.8) in the above equation and rearranging we finally obtain

$$\ddot{h_2}+\left[\frac{1}{A_2}\left(\frac{1}{A_1}+\frac{1}{R_2}\right)+\frac{1}{A_1R_1}\right]\dot{h_2}-\left[\frac{1}{A_1A_2R_1^2}-\frac{1}{A_1A_2R_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]h_2=\frac{1}{A_1A_2R_1}q$$
$$(1.11)$$

Equation (1.11) is a differential equation of order 2 of the form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0 x$$

In general, it is possible to write n simultaneous first order differential equations, each representing the dynamic behavior of a 'simple' system as an n^{th} order differential equation representing the dynamic behavior of the resulting 'complex' system which is a collection of the interacting subsystems [31].

1.3 Classification of Control Systems

Based on some parameters, we can classify the control systems into the following ways:

1-3-1 Continuous time and Discrete-time Control Systems:

Control Systems can be classified as continuous time control systems and discrete time control systems based on the type of the signal used. In continuous time control systems, all the signals are continuous in time. But, in discrete time control systems there exists one or more discrete time signals.

1-3-2 SISO and MIMO Control Systems:

Control Systems can be classified as SISO control systems and MIMO control systems based on the number of inputs and outputs present.

Definition1) SISO (Single Input and Single Output): control systems have one input and one output. in the radio it use of only one antenna both in the transmitter and receiver.

MIMO (Multiple Inputs and Multiple Outputs): systems have more than one input and more than one output an antenna technology for wireless communications in which multiple antennas are used at both the source (transmitter) and the destination (receiver).

1-3-3 Open Loop and Closed Loop Control Systems:

Control Systems can be classified as open loop control systems and closed loop control systems based on the feedback path. In open loop control systems, output is not fed-back to the input. So, the control action is independent of the desired output. The following figure1.8 shows the block diagram of the open loop control system.

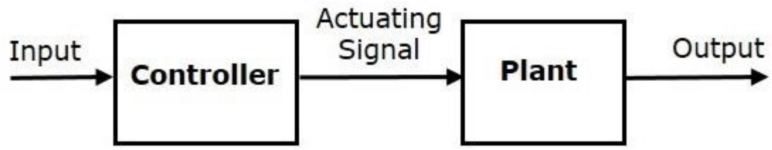


Figure1.8: open loop control system

Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signal is given as an input to a plant or process which is to be controlled. So, the plant produces an output, which is controlled. The traffic lights control system which we discussed earlier is an example of an open loop control system.

In closed loop control systems, output is fed back to the input. So, the control action is dependent on the desired output. The following figure1.9 shows the block diagram of negative feedback closed loop control system.

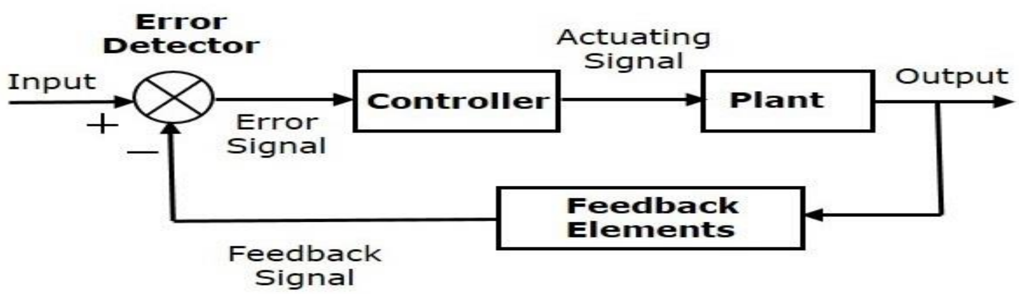


Figure1.9: closed loop control system

The error detector produces an error signal, which is the difference between the input and the feedback signal. This feedback signal is obtained from the block (feedback elements) by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

So, the controller produces an actuating signal which controls the plant. In this combination, the output of the control system is adjusted automatically till we get the desired response. Hence, the closed loop control systems are also called the automatic control systems. Traffic lights control system having sensor at the input is an example of a closed loop control system. The differences between the open loop and the closed loop control systems are mentioned in the following table1.

Open Loop Control Systems	Closed Loop Control Systems
Control action is independent of the desired output.	Control action is dependent of the desired output.
Feedback path is not present.	Feedback path is present.
These are also called as non-feedback control systems.	These are also called as feedback control systems.
Easy to design.	Difficult to design.
These are economical.	These are costlier.
Inaccurate.	Accurate.

table1: The differences between the open loop and the closed loop control systems

1-4 Feedback

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then it is known as feedback. Feedback plays an important role in order to improve the performance of the control systems. In this chapter, let us discuss the types of feedback and effects of feedback.

1-4-1 Positive Feedback

The positive feedback adds the reference input, $R(s)$ and feedback output. The following figure1.10 shows the block diagram of positive feedback control system.

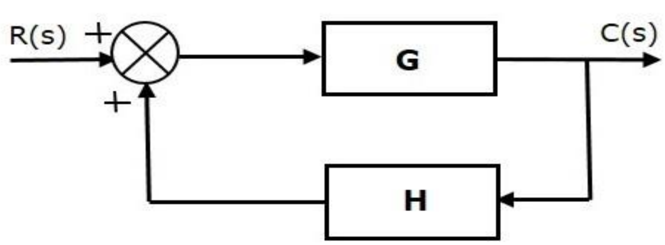


Figure1.10: the block diagram of positive feedback control system

For the time being, consider the transfer function of positive feedback control system is,

$$T = \frac{G}{1-Gh} \qquad (1.12)$$

Where,

- T is the transfer function or overall gain of positive feedback control system.
- G is the open loop gain, which is function of frequency.
- H is the gain of feedback path, which is function of frequency.

1-4-2 Negative Feedback

Negative feedback reduces the error between the reference input, $R(s)$ and system output. The following figure1.11 shows the block diagram of the negative feedback control system.

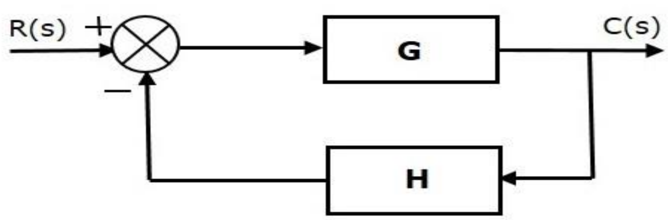


Figure1.11: the block diagram of the negative feedback control system

Transfer function of negative feedback control system is,

$$T = \frac{G}{1+Gh} \qquad (1.13)$$

Where

- T is the transfer function or overall gain of negative feedback control system.
- G is the open loop gain, which is function of frequency.
- H is the gain of feedback path, which is function of frequency.

1-4-3 Effects of Feedback

Let us now understand the effects of feedback. Effect of Feedback on Overall Gain

From Equation 1.13, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and $(1 + GH)$. So, the overall gain may increase or decrease depending on the value of $(1 + GH)$.

- If the value of $(1+GH)$ is less than 1, then the overall gain increases. In this case, 'GH' value is negative because the gain of the feedback path is negative.
- If the value of $(1+GH)$ is greater than 1, then the overall gain decreases. In this case, $'GH'$ value is positive because the gain of the feedback path is positive.

In general, $'G'$ and $'H'$ are functions of frequency. So, the feedback will increase the overall gain of the system in one frequency range and decrease in the other frequency range.

1-4-4 Effect of Feedback on Sensitivity

Sensitivity of the overall gain of negative feedback closed loop control system (T) to the variation in open loop gain (G) is defined as

$$S_G^T = \frac{\frac{\partial T}{T}}{\frac{\partial G}{G}} = \frac{percentage\ change\ in\ T}{percentage\ change\ in\ G} \qquad (1.14)$$

Where, ∂T is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} \qquad (1.15)$$

Differentiating partially differentiation with respect to G on both sides of

Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \qquad (1.16)$$

From equation 2, you will get

$$\frac{G}{T} = 1+GH \qquad (1.17)$$

Substitute Equation (1.14)and equation (1.17) in equation (1.15)

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So, we got the sensitivity of the overall gain of closed loop control system as the reciprocal of $(1+GH)$. So, Sensitivity may increase or decrease depending on the value of $(1+GH)$.

- If the value of $(1 + GH)$ is less than 1, then sensitivity increases. In this case, 'GH' value is negative because the gain of feedback path is negative.
- If the value of $(1 + GH)$ is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, feedback will increase the sensitivity of the system gain in one frequency range and decrease in the other frequency range. Therefore, we have to choose the values of 'GH' in such a way that the system is insensitive or less sensitive to parameter variations.

1-4-5 Effect of Feedback on Stability

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. In equation 1.13, if the denominator value is zero ($i.e., GH = -1$), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore, we have to properly choose the feedback in order to make the control system stable.

1-4-6 Effect of Feedback on Noise

To know the effect of feedback on noise, let us compare the transfer function relations with and without feedback due to noise signal alone. Consider an open loop control system with noise signal as shown figure1.12.

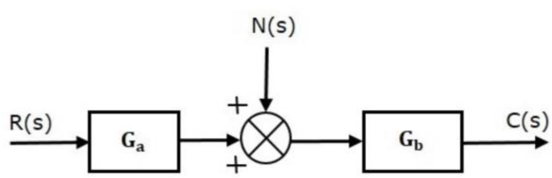


Figure1.12: open loop control system with noise signal

The open loop transfer function due to noise signal alone is

$$G_b = \frac{C(s)}{N(s)} \qquad (1.18)$$

It is obtained by making the other input $R(s)$ equal to zero. Consider a closed loop control system with noise signal as shown figure1.13.

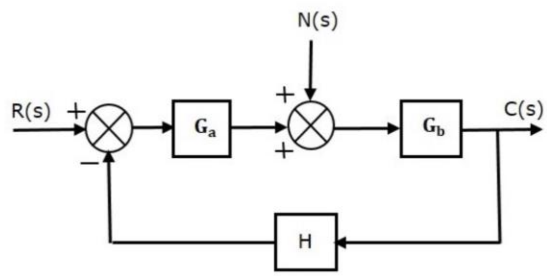


Figure1.13: closed loop control system with noise signal

The closed loop transfer function due to noise signal alone is

$$\frac{C(s)}{N(s)} = \frac{G_b}{1+G_aG_bH} \qquad (1.19)$$

It is obtained by making the other input $R(s)$ equal to zero. Compare Equation $(1.18\ and\ 1.19)$ in the closed loop control system, the gain due to noise signal is decreased by a factor of $(1 + G_aG_bH)$ provided that the term $(1 + G_aG_bH)$ is greater than one.

1-5 Transfer functions

An input-output description of a system is essentially a table of all possible input-output pairs. For linear systems the table can be characterized by one input pair only, for example the impulse response or the step response the transfer function of a two-port electronic circuit like an amplifier might be a two-dimensional graph of the scalar voltage at the output as a function of the scalar voltage applied to the input; the transfer function of an electromechanical actuator might be the mechanical displacement of the movable arm as a function of electrical current applied to the device; the transfer function of a photodetector might be the output voltage as a function of the luminous intensity of incident light of a given wavelength. this section we will consider Transfer Function of a Linear Ordinary differential equations ODE [36]. In general form the n^{th} order system having a single input and a single output (figure1.14) may have an associated differential equation

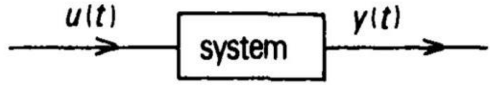


Figure1.14

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u \qquad (1.20)$$

where u is the input and y is the output. Note that here we have generalized our previous system description to allow both the input and its derivatives to appear. The differential equation is completely described by two polynomials

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n \qquad (1.21)$$

$$b(s) = b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots b_{n-1} s + b_n$$

where the polynomial $a(s)$ is the characteristic polynomial of the system.

To determine the transfer function of the system (1.20), let the input be

$u(t) = e^{st}$. Then there is an output of the system that also is a exponential function $y(t) = y_0 e^{st}$. Inserting the signals in (1.20)we find

$$(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n) y_0 e^{st}$$
$$= (b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots b_{n-1} s + b_n) e^{-st}$$

IF $a(s) \neq 0$ IF follows that

$$y(t) = y_0 e^{st} = \frac{b(s)}{a(s)} e^{st} = G(s) u(t) \qquad (1.22)$$

The transfer function for this system is thus the rational function

$$G(s) = \frac{b(s)}{a(s)} \qquad (1.23)$$

where the polynomials $a(s)$ and $b(s)$ are given by (1.21). Notice that the

transfer function for the system (1.20) can be obtained by inspection, since the coefficients of $a(s)$ and $b(s)$ are precisely the coefficients of the derivatives u and y [31].

Equation (1.20) and (1.23)can be used to compute the transfer function of many simple ODEs. the following table2 gives some of the more common forms:

Type	ODE	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s

First order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_n\dot{y} = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
PID controller	$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

Table2: some of the more common forms

Example 1.1

Find the transfer function for a fluid tank having a cross-sectional area A with a head $h(t)$, one inflow $q(t)$, It also contained an outflow valve containing resistance A.

Solution:

The system can be described by the differential equation (1.7)

$$A\frac{dh}{dt} = q - \frac{1}{R}h$$

$\Longrightarrow AR\frac{dh}{dt} + h = qR$ [where A and R are constants]

By taking the Laplace transforms of the above equation with an initial condition of zero.

We get

$$ARsH(s) + H(s) = Q(s)R$$

Hence

$$G(s) = \frac{H(s)}{Q(s)} = \frac{R}{ARs + 1}$$

Example 1.2

Find the transfer function of the system described by the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2u$$

Where $u(t)$the input of system and $y(t)$ the output of the system.

Solution:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2u$$

By taking the Laplace transforms of both sides of the equation (assuming zero initial conditions), we obtain

$$s^2Y(s) + 6sY(s) + 8Y(s) = 2U(S)$$

$$\Longrightarrow (s^2 + 6s + 8)Y(s) = 2U(S)$$

Hence

$$G(s) = \frac{2}{(s+2)(s+4)}$$

$1-6$ State Space Model

Is a mathematical model in control theory. It is a state-space representation of a physical system of a set of inputs and outputs along with some set of state variables related by first-order differential equations. State variables in this model are a type of variable whose value changes over time and depends on the values that have been given for the input variables. The value of the output variables depends on the value of the state and input variables. Putting a model into state-space representation is the basis for many methods in control analysis and the dynamics process. A system characterized by an n^{th} order differential equation. We shall also assume that the systems we are dealing with are autonomous, which implies that the free system (where the input is zero) does not depend explicitly on time.

The system equation has the form:

$$\frac{d^ny(t)}{dt^n} + a_1\frac{d^{n-1}y(t)}{dt^{n-1}} + a_2\frac{d^{n-2}y(t)}{dt^{n-2}} \dots + a_{n-1}\frac{dy(t)}{dt} + a_ny(t) = u(t) \quad (1.24)$$

We know that:

$$\dot{y} = \frac{dy(t)}{dt}, \ddot{y} = \frac{d^2y(t)}{dt^2}, \dots, y^{(n)} = \frac{d^ny(t)}{dt^n}$$

Then the (1.24) has the form:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}\dot{y}(t) + a_n y(t) = u(t) \qquad (1.25)$$

It is assumed that:

$y(0), \dot{y}(0), \ddot{y}(0), \ldots, y^{(n-2)}(0), y^{(n-1)}$ are knowns.

Introduce the following change of variables

$$x_1 = y(t), x_2 = \dot{y}(t), x_3 = \ddot{y}(t), \ldots, x_{n-1} = y^{(n-2)}(t), x_n = y^{(n-1)}(t)$$

which after taking derivatives leads to

$$\frac{dx_1(t)}{dt} = \dot{x}_1 = \frac{dy(t)}{dt} = x_2(t) \Longrightarrow \dot{x}_1 = x_2(t)$$

$$\frac{dx_2(t)}{dt} = \dot{x}_2 = \frac{d^2y(t)}{dt^2} = x_3(t) \Longrightarrow \dot{x}_2 = x_3(t)$$

$$\frac{dx_3(t)}{dt} = \dot{x}_3 = \frac{d^3y(t)}{dt^3} = x_4(t) \Longrightarrow \dot{x}_3 = x_4(t)$$

$$\vdots$$

$$\frac{dx_{n-1}(t)}{dt} = \dot{x}_{n-1} = \frac{d^{n-1}y(t)}{dt^{n-1}} = x_n(t) \Longrightarrow \dot{x}_{n-1} = x_n(t)$$

And from (1.25)

$$\dot{x}_n = -a_n x_1(t) - a_{n-1}x_2(t) - a_{n-2}x_3(t) - \cdots - a_1 x_n(t) + u(t)$$

It can be written as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

(1.26)

That is as

$$\dot{x} = Ax(t) + Bu(t)$$

Where The components of x are the state variables x_1,$x_2, \ldots x_n$. They can be considered as the coordinate's axes of an n-dimensional space called the state space , A and B matrices and the output of the system is y,

which was defined as $x_1(t)$ above, and is written in matrix form as

$$y(t) = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \qquad (1.27)$$

That is as

$$y(t) = Cx(t)$$

Where

$$C = [1 \quad 0 \quad \cdots \quad 0]$$

The combination of equations (1.26) and (1.27) in the form

$$\dot{x} = Ax(t) + Bu(t) \qquad (1.28)$$

$$y(t) = Cx(t)$$

The system (1.28) are known as the state equations, The matrix A in equation (1.26) is said to be in companion form.

Example 1.3

Consider two forms of state equations of the system defined by:

$$y^{(6)} - 4y^{(5)} + 3y^{(4)} + y^{(3)} + 6y^{(1)} - 2y = 3u$$

Solution:

Using the state variables defined above, we have

$$x_1(t) = y\,, x_2(t) = y^{(1)}, x_3(t) = y^{(2)}, x_4(t) = y^{(3)}, x_5(t) = y^{(4)}, x_6(t) = y^{(5)}$$

Then

$\dot{x}_1 = x_2(t)$

$\dot{x}_2 = x_3(t)$

$\dot{x}_3 = x_4(t)$

$\dot{x}_4 = x_5(t)$

$\dot{x}_5 = x_6(t)$

$\dot{x}_6 = 2x_1(t) - 6x_2(t) - x_4(t) - 3x_5(t) + 4x_6(t) + 3u(t)$

Hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -6 & 0 & -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} u(t)$$

And

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$$

Chapter two

Controllability and Observability

2-1 introduction

Controllability and control are two basic and important concepts in modern control theory. These two concepts were defined by Kalman in 1960[37] with the aim of identifying the extent to which the system can be monitored and controlled. The fundamental questions to be answered for a control system, in particular for a multivariable system are:

a) Can a control function $u(t)$ be found to transform the system's initial state x_t into the desired final state x_f in finite time?

b) Can the state of the system be determined by measuring its performance over a finite time interval?

The two concepts involved are called controllability and observability. So, if the answer to the first question is "yes", then the system is controllable. If the answer to the second question is yes, then the system is also observable. It should be recognized that these issues are fundamental. For example, unless the system is controllable, it does not make sense to try to control a system by feedback of a state variable that allows the poles of the system to be positioned arbitrarily. Likewise, unless the system is observable, there is no point in trying to reconstruct unmeasurable state variables of the system through devices called observers. Controllability and observability are actually two dual concepts, which are closely related to the cancellation of poles and zeros in the system transfer function.

2-2 Controllability

We say about a system that it is controllable if and only if it is possible through the control vector to bring the system from the initial state $x(t_0) = x_0$ to any final state $x(t_f) = x_f$ within a specified time $t > 0$.

In the case of nonlinear systems, these equations take the following form:

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

$$y = C(t)x(t) + D(t)u(t)$$

In the case of linear systems fixed with time, the equations take the following form:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y = Cx(t) + Du(t)$$

Where

A is the state matrix of order $n \times n$

B is control matrix of order $n \times m$

C is the output matrix of order $1 \times n$

D is Direct transfer matrix of order $1 \times m$

n: The number of state vector state.

m: The number of control vector state.

We consider a system described by the state equations:

$$\dot{x} = Ax(t) + Bu(t) \qquad (2.1)$$

$$y = Cx(t)$$

With the transformation:

$$x(t) = pz(t) \qquad (2.2)$$

we can transform equation (2.1) into the form:

$$\dot{z} = A_1 z(t) + B_1 u(t) \qquad (2.3)$$

$$y = C_1 z(t)$$

Where $A_1 = P^{-1}AP$, $B_1 = P^{-1}B$ and $C_1 = CP$. Assuming that A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ we can choose P so that A_1 is a diagonal matrix, that is,

$$A_1 = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

If $n = m = 2$, the first of the equations (2.3) has the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Which is written as

$$\dot{z}_1 = \lambda_1 z_1 + b_1^T u \qquad (2.4)$$

$$\dot{z}_2 = \lambda_2 z_2 + b_1^T u$$

Where b_1^T and b_2^T are the row vectors of the matrix B_1

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Which can be written as

$y_1 = c_{11} z_1 + c_{12} z_2$

$y_2 = c_{21} z_1 + c_{22} z_2$

Or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} z_1 + \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} z_2$$

So that

$y = c_1 z_1 + c_2 z_2$ (2.5)

where c_1 and c_2 are the column vectors of C_1. So, in general, equation (2.3) can be written in the form:

$$\dot{z}_i = \lambda_i z_i + b_i^T u(t) \qquad (i = 1,2,3,\dots,n)$$

$$y = \sum_{i=1}^{n} c_i z_i \qquad (2.6)$$

It is seen from equation (2.6) that if b_i^T the i^{th} row of B_1 has all zero components, then

$$\dot{z}_i = \lambda_i z_i + 0$$

and the input $u(t)$ has no influence on the i^{th} mode of the system. The mode is said to be uncontrollable, and a system having one or more such modes is uncontrollable [31,32].

Example 2.1

Check whether the system having the state-space representation

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is controllable?

Solution:

The characteristic equation is

$|\lambda I - A| = \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = 1 \;\&\; \lambda = 2$

The corresponding eigenvectors are

$x_1 = [1 \quad 1]^T \; and \; x_2 = [2 \quad 3]^T$

so that the modal matrix is

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad and \; P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [-1 \quad -4] z$$

This equation shows that the first mode is uncontrollable and so the system is uncontrollable.

Example 2.2

Check whether the system having the state-space representation

$$\dot{x} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is controllable?

Solution:

The characteristic equation is

$|\lambda I - A| = \lambda^2 - 1 = 0$

$(\lambda - 1)(\lambda + 1) = 0$

Then $\lambda = 1$ & $\lambda = -1$

The corresponding eigenvectors are

$x_1 = [1 \quad 1]^T$ *and* $x_2 = [2 \quad 3]^T$

so that the modal matrix is

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Then the system is controllable.

On the basis of the above result, we now derive an extremely useful criterion for determining whether a system is controllable. Although at this stage we consider only the necessity of this criterion, it is also a sufficient condition. To simplify the notation and the mathematical manipulations, we consider a SISO (single-input and single-output) system, so that in equation (2.1) B is a one column matrix, that is a column vector b (say), and C is a row vector$\acute{c}_1$. The result holds for the more general case when the system is multivariable.

Equations (2.1) and (2.3) are then written as

$$\dot{x} = Ax(t) + bu(t) \qquad (2.1\ a)$$

$$y = Cx(t)$$

And

$$\dot{z} = A_1 z(t) + b_1 u(t) \qquad (2.3\ a)$$

$$y = \acute{c}_1 z(t)$$

We have chosen an indirect way of deriving the controllability criterion. It has the advantage of simplicity, but a penalty we pay for this is some loss in the logic behind the setting up of the criterion.

We have established that the necessary condition for the system defined by equation (2.1 a) to be controllable is that the components of the vector $b_1 = [\beta_1 \quad \beta_2 \quad \dots \quad \beta_n]^T$ in equation (2.3 a)are all non-zero

In equation (2.3 a) the matrix $A_1 = diag\{\lambda_1, \lambda_2. \dots, \lambda_n\}$where the eigenvalues $\lambda_1, \lambda_2. \dots, \lambda_n$ are assumed distinct.

Hence the matrix:

$$\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{bmatrix}$$

Has linearly independent columns, so that it is non-singular. It follows that the necessary condition to be controllable is that the (partitioned) matrix:

$$Q_p = [b_1 \quad A_1 b_1 \quad A_1^2 b_1 \quad \cdots \quad A_1^{n-1} b_1] = \begin{bmatrix} \beta_1 & \lambda_1\beta_1 & \dots & \lambda_1^{n-1}\beta_1 \\ \beta_2 & \lambda_2\beta_2 & \dots & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \lambda_n\beta_n & \dots & \lambda_n^{n-1}\beta_n \end{bmatrix} \qquad (2.7)$$

The matrix $Q_p$is non-singular.

Since

$$A_1 = P^{-1}AP \; and \; b_1 = P^{-1}b$$

We have

$A_1 b_1 = P^{-1}AP\, P^{-1}b = P^{-1}Ab$

$A_1^2 b_1 = P^{-1}A^2P\, P^{-1}b = P^{-1}A^2b$

$\vdots$

$A_1^{n-1} b_1 = P^{-1}A^{n-1}P\, P^{-1}b = P^{-1}A^{n-1}b$

So that

$$Q_p = P^{-1}[b \quad Ab \quad A^2B \quad \cdots \quad A^{n-1}b] = P^{-1}Q_c$$

Where

$$Q_c = [b \quad Ab \quad A^2B \quad \cdots \quad A^{n-1}b] \qquad (2.8)$$

Since Q_p(for a controllable system) and P^{-1} are both non-singular, Q_c (for a controllable system) is also non-singular.

As Q_p is non-singular, its n columns are linearly independent. So that the rank of the matrix Q_c written as $r(Q_c)$ is n.

Example 2.3

Let's solve the previous example (2.1**)** in this way

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u$$

$$y = [1 \quad -2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 6 \end{bmatrix} and \; b = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

So that

$$r(Q_c) = r[b \quad Ab] = r \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix} = 1$$

The rank of Q_c is less than 2, the system is uncontrollable.

2-2-1 Controllability test

Let us know a simple method by which we can find out the controllability of a system, consider a system described by the state equations:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y = Cx(t)$$

Step 1: We write the matrix Q_c(*is called the system controllability matrix*)

$$Q_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

Step 2: We Find the determinant of matrix Q_c if it is not equal to zero then the control system is controllable or if determinant of Q_c equal to zero then the control system is uncontrollable.

Example 2.4

Let's solve the previous example(example1**)** in this way

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \; and \; B = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Step 1: We write the matrix Q_c

$$Q_c = \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix}$$

Step 2: We find the determinant of matrix Q_c

$$|Q_c| = (4 * 12) - (8 * 6) = 0$$

then system is uncontrollable.

Example 2.5

Verify the controllability of control system which is the presented by state equation:

$\dot{x} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

Solution:

Given $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \; and \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , n = 2$

Step 1: $Q_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = [B \quad AB] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Step 2:$|Q_c| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

the system is controllable.

The matrix A is said to be order higher than 3×3,if is difficult to know whether the system is controllable or not in the previous way, so we assume that

$$\Phi = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-m}B]$$

When m is number of inputs

Step 1: write the matrix $Q_c($ *is called the system controllability matrix*$)$

$$\Phi = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-m}B]$$

Step 2: find rank of Φ if it is equal to n then the control system is controllable or if rank of Φ not equal to n then the control system is uncontrollable.

Or we can calculate the determinant $\Phi\Phi^T$ if it is not equal to zero then the control system is controllable or if determinant of $\Phi\Phi^T$ equal to zero then the control system is uncontrollable.

Example 2.6

Verify the controllability of control system which is the presented by state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

Solution:

Step 1: $\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & -4 \end{bmatrix}$

Step 2: $rank(\Phi) = 4 = \text{n}$

Then the system is controllable.

2-2-2 Program for finding controllability

A MATLAB program was designed to find controllability and was saved under the name “ctrb” It is used when we need it, as shown below:

$Function\ co = ctrb(A, B)$

$N = length(A);$

$co = ctrb(a, b);$

$if\ rank(co) \cong n$

$disp('no\ contrable')$

$else$

$disp('\ contrable')$

end

end

Example 2.7

Is the system given as follows controllable?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

Using the preceding method results in the following

$\gg A = [-1 \quad 0 \quad 0; -1 \quad -2 \quad 0; 1 \quad 0 \quad 0];$

$\gg B = [1; 0; 0];$

$\gg co(A, B)$

$contable$

$ans =$

$$\begin{matrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -1 \end{matrix}$$

That is, the system is controllable and the value of the matrix Q_c:

$$Q_c = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Example 2.8

Check if the system below is controllable

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ x_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u$$

$$y(t) = \begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Solution:

Using the preceding method results in the following

$\gg A = [1 \quad 0 \quad 1; 0 \quad 1 \quad 0; 1 \quad 1 \quad 1];$

$\gg B = [1; 0; 1];$

$\gg co(a, b)$

$not\ contable$

$ans =$

$$\begin{matrix}1 & 2 & 4\\ 0 & 0 & 0\\ 1 & 2 & 4\end{matrix}$$

That is, the system is uncontrollable and the value of the matrix Q_c:

$$Q_c = \begin{bmatrix}1 & -1 & 1\\ 0 & -1 & 3\\ 0 & 1 & -1\end{bmatrix}$$

2-3 Observability

A system is said to be observable if the initial vector $x(t)$ can be found from the measurement of $u(t)$ and $y(t)$. The plant described by (2.1) is completely state observable if the inverse matrix exists [34].

By using the transform $x = Pz(x)$ as in the (2-1) section, we end up with the system state equations in the form of equation (2.6), that is

$$\dot{z} = A_1 z(t) + B_1 u(t)$$

$$y = C_1 z(t)$$

If a row of the matrix C_1 is zero, the corresponding mode of the system will not appear in the output y. In this case the system is unobservable, since we cannot determine the state variable corresponding to the row of zeros in C_1 from y.

Example 2.9
Check whether the system having the state-space representation

$$\dot{x} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is observable?
Solution:
The characteristic equation is
$|\lambda I - A| = \lambda^2 - 1 = 0$
$(\lambda - 1)(\lambda + 1) = 0$
$\Rightarrow \lambda = 1 \;\&\; \lambda = -1$
The corresponding eigenvectors are
$x_1 = [1 \quad 1]^T$ *and* $x_2 = [2 \quad 3]^T$
so that the modal matrix is
$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$
Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Then the system is unobservable.

Example 2.10
Check whether the system in (example1) is observable?
Solution:
The characteristic equation is
$|\lambda I - A| = \lambda^2 - 3\lambda + 2 = 0$
$(\lambda - 1)(\lambda - 2) = 0$
$\Rightarrow \lambda = 1 \;\&\; \lambda = 2$
The corresponding eigenvectors are
$x_1 = [1 \quad 1]^T$ *and* $x_2 = [2 \quad 3]^T$
so that the modal matrix is
$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & -4 \end{bmatrix} z$$

Then the system is observable.

The two example (2.9-2.10) illustrates the importance of the observability concept. In this case we have unstable system, whose instability is not observed in the output measurement. The dual controllability concept is of equal theoretical importance. An uncontrollable system has one or more modes which are not influenced by the input.

We now derive a criterion for observability in a similar manner to that used to derive the controllability criterion.

Again, for simplicity we consider a $SISO$ system, but the result holds for the more general multivariable system. We have seen that the necessary conditions for systems defined by equation (2.1 a) to be observable is that the components of the vector$b_1 = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n]^T$ in equation $(2.3\ a)$ are all non-zero for a controllable system we have the matrix

$$Q_1 = \begin{bmatrix} c_1^T \\ c_1^T A_1 \\ \vdots \\ c_1^T A_1^{n-1} \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_2\lambda_1 & \gamma_2\lambda_2 & \cdots & \gamma_n\lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n\lambda_1^{n-1} & \gamma_2\lambda_2^{n-1} & \ldots & \gamma_n\lambda_n^{n-1} \end{bmatrix} \qquad (2.9)$$

The matrix $Q_1$is non-singular.

Since

$$A_1 = P^{-1}AP \ and \ C_1^t = c^T P$$

We have

$$c_1^T A_1 = c^T PP^{-1}AP = c^T AP$$
$$A_1^2 b_1 = c^T PP^{-1}A^2 P = c^T A^2 P$$
$$\vdots$$
$$A_1^{n-1} b_1 = c^T PP^{-1}A^{n-1} P = c^T A^{n-1} P$$

So that

$$Q_1 = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} P = Q_o P$$

Where

$$Q_o = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} \qquad (2.10)$$

Since Q_o(for a observable system) and P are both non-singular, Q_o (for a observable system) is also non-singular.

2-3-1 The observability criterion

In equation (2.10) if the rank of matrix Q_o is n, the system can be called observable system. If rank Q_o less than n ,the system is un observable.

Example 2.11

Check whether the system in (example 2.2) is observable?

Solution:

$$\mathrm{c}^{\mathrm{T}} = [3 \quad -2], \mathrm{A} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$$

Hence

$$Q_o = \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix}$$

So that

$$r(Q_o) = 1$$

It follows that the system is unobservable.

2-3-2 Observability test

A control system is said to be observable if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

To find out whether the control system is observable or not, we use a Kalman's test:

Step 1: form the matrix $Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T]$

Step 2: Take determinant of Q_o if it is not equal to zero then the control system is observable or if determinant of Q_c equal to zero then the control system is not observable [4].

Example 2.12

Verify the observability of control system which is the presented by state equation:

$$\dot{x} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

Given $A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $and\ c = [1 \quad 1]$ $, n = 2$

Step 1: $Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T \quad \ldots \quad (A^T)^{n-1}C^T] = [C^T \quad A^TC^T] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$

Step 2: $|Q_o| = \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$

the system is observable.

2-3-3 Program for finding observability [35]

A MATLAB program was designed to find observability and was saved under the name “obsv” It is used when we need it, as shown below:

```
function ob = obsv(A,C)
%'The function ob = obsv(A,C)returns the transformation matrix '
%' ob = [C; CA; CA^2; ...CA^(n-1)].The system is completely state'
%' observable if and only if o has a rank of n.
n = length(A);
for i = 1:n;
o(n+1-i,:) = C * A^(n-i);
end
if rank(ob)~ = n
disp('System is not state observable')
else
disp('System is state observable')
end
```

Example 2.13

Check whether the system having the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u$$

$$y(t) = [1 \quad 1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observability?

Solution:

$>> \ A = [0\ 1\ 0; 0\ 0\ 1; -6\ -11\ -6];$
$>> \ C = [1\ 1\ 1];$
$>> \ ob(a, c)$
$System\ is\ state\ observable$
$ans\ =$

1	1	1
−6	−10	−5
30	49	20

That is, the system is $observable$ and the value of the matrix Q_o:

$$Q_o = \begin{bmatrix} 1 & 1 & 1 \\ -6 & -10 & -5 \\ 30 & 49 & 20 \end{bmatrix}$$

Example 2.14
A MATLAB program was designed to find Controllability and Observability for the system given as follows:

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:
$A = [-2 \quad 0; -1 \quad -1];$
$B = [1; \quad 1];$
$C = [0 \quad 1];$
%
% $(a)\ Determine\ the\ eigenvalues$
$eig(A)$
$ans =$
-1
-2
%
% $(b)\ Determine\ the\ transfer\ function$
$sys = \ ss(A, B, C, 0);$
$sys = tf(sys)$

$Transfer\ function$:

$$\frac{s+1}{s\hat{}2 + 3\,s + 2} = \frac{s+1}{(s+1)(s+2)}$$

$sys = minreal(sys)$

$Transfer\ function:$

$\frac{1}{s+1}$

%

% (c) $Determine\ controllability\ and\ observability$

%

% The "$ctrb$" $function\ computes\ the\ controllability\ matrix$

$co = ctrb(A, B)$

$co =$

$$\begin{matrix} 1 & -2 \\ 1 & -2 \end{matrix}$$

$rank(co)$

$ans =$

% $The\ number\ of\ uncontrollable\ states$

$uncon = length(A) - rank\ (co)$

$uncon =$

$$1$$

%

% The "$obsv$" $function\ computes\ the\ observability\ matrix$

$ob = obsv(A, C)$

$ob =$

$$\begin{matrix} 0 & 1 \\ -1 & -1 \end{matrix}$$

$rank\ (ob)$

$ans =$

$$2$$

%

$\% \ (d) \ MATLAB's \ version \ shows \ one \ controllable \ state \ [Ab, Bb, Cb,$

$Db, T]$ ¼ $CANON \ (A, B, C, 0, 'modal')$

$Ab =$

$$\begin{matrix} -1 & 0 \\ 0 & -2 \end{matrix}$$

$Bb =$

$$\begin{matrix} 0 \\ 1.4142 \end{matrix}$$

$Cb =$

$$\begin{matrix} 1.0000 & 0.7071 \end{matrix}$$

$Db =$

$$0$$

$T =$

$$\begin{matrix} -1.0000 & 1.0000 \\ 1.4142 & 0 \end{matrix}$$

$T * A * \ inv \ (T)$

$ans =$

$$\begin{matrix} -1 & 0 \\ 0 & -2 \end{matrix}$$

2-4 Decomposition of System State

From the discussion in the previous two sections, it is clear that the state variables (or, equivalently, the corresponding modes) of a linear system can generally be divided into the following four exclusive groups:

Case 1: Controllable and Observable

Case 2: Controllable but unobservable

Case 3: Uncontrollable but observable

case 4: Uncontrollable and unobservable.

Assuming that the system matrix A has different eigenvalues, the state equation can be simplified to the following form by appropriate transformation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}A_1 & 0 & 0 & 0\\ 0 & A_2 & 0 & 0\\ 0 & 0 & A_3 & 0\\ 0 & 0 & 0 & A_4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\\ 0\\ 0\end{bmatrix}u \qquad (2.11)$$

$$y = [C_1 \quad 0 \quad C_3 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

The (transformed) system matrix A is put in "block diagonal" form, with each $A_i(i = 1,2,3,4)$ having a diagonal form. The suffix i of the state variable vector x means that the elements of this vector are the state variables corresponding to the i^{th} case defined above.

Example 2.15

Classify the state variables in a system defined by the following state equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix} + \begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\\ 0 & 0\\ 1 & 0\\ 0 & 2\end{bmatrix}u$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & 0 & -1 & 2 & 0 & 1\\ 1 & 0 & 1 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix}$$

Solution:

By inspection we can classify the state variables into the four groups as follows:

Case1: Controllable and observable, x_1, x_3 *and* x_6.

Case 2: Controllable and unobservable x_5.

Case 3: Uncontrollable and observable x_4.

Case 4: Uncontrollable and unobservable x_2.

We can represent the decompositions of the state variables into four groups

by a diagram (see Figure2.1) showing the system divided into four subsystems each

having state variables belong to one group only as indicated by the suffix i of

S_i.

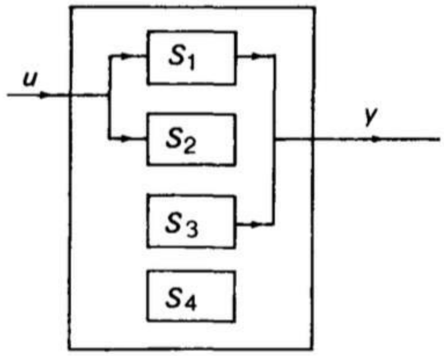


Figure2.1: The system divided into four subsystems

This insight into the system structure explains the difference which may exist between the form of the system transfer function calculated from the system differential equations and that obtained by experimentation (that is, by obtaining the system frequency response.

We define the transfer function($G(s)$ as $G(s) = \frac{Y(s)}{U(s)}$ or see equation (1.23)) as the ratio of the Laplace transform of the output $Y(s)$ to the input $U(s)$. The transfer function obtained from the differential equation (or equivalently from the system equation of state) includes all state variables (or modes) of the system. But the transfer function discovered through experimentation involves the part of the system that is affected by the input and affects the output. It can be seen from figure 2.1 that the transfer function of the subsystem Si is determined and only includes controllable and observable state variables (or modes).

In general, the transfer function $G(s)$ represents only the subsystem s_1 of the considered system, and indeed on adding to s_1 the subsystems S_2 , S_3 and S_4 has no effect on $G(s)$.

Chapter three

Stability of linear control system

3-1 introduction

The Open and closed-loop control system transfer functions have certain basic characteristics that permit transient and steady-state analyses of the feedback-controlled system. five factors of prime importance in feedback-control systems are stability, the existence and magnitude of the steady-state error, controllability, observability, and parameter sensitivity. the stability is possibly the most important consideration when designing a control system. The problems involved are not only important but extremely complicated - indeed much present-day research in control is concerned with stability. That is not to say that methods for determining stability do not exist-on the contrary many methods do exist. Unfortunately, they deal only with certain aspects of stability. For nonlinear systems and even for linear systems it is not always possible to determine analytically for what ranges of the relevant parameters the system remains stable. The difficulties begin with the definitions of stability-, we all know roughly what is involved, but it is very difficult to cover all possible cases. Perhaps a suitable analogy is the definition of probability - we all know roughly what is meant, but its definition is very 'theoretical' and is difficult to interpret in practical situations.

3-2 The concept of stability

The stability of the system is to make the system equilibrium which does not accept the major and sudden changes that leads the system to be in a state of dispersion or may cause noise, which hinders the system from well-functioning. Before we give a mathematical definition of stability, let us explain this concept with examples from our daily life such as calmness after a big challenge, a calm, stable life where you don't have unruly ups and downs. The ability of an object, such as a ship or plane, to maintain balance or resume its original upright position after its displacement, such as the sea or strong winds, the state or quality of existence.

Mathematically, the system is considered stable when the output is within certain values, meaning and does not tends to infinity, the researchers need to define the transfer function as shown in Figure3.1 [38].

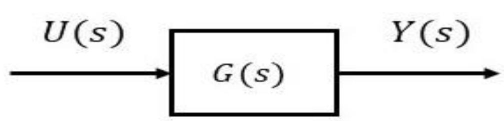


Figure 3.1

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

When $n^{th}$is of the system ($n \geq m$) and $D(s)$ is characteristic polynomial when the $D(s)$ When getting close to zero ($D(s) = 0$) shown below

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (3.1)$$

are called the characteristic equation.

The roots of the numerator polynomial

i.e., $N(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 = 0 \qquad (3.2)$

are called the zeroes of the system.

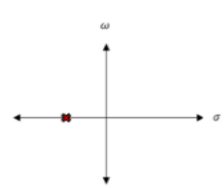
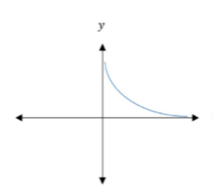
The roots of the denominator polynomial equation (3.1) are called poles of the system.

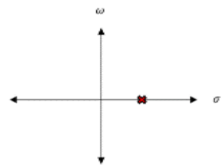
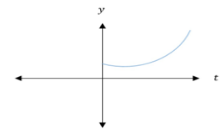
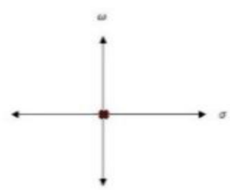
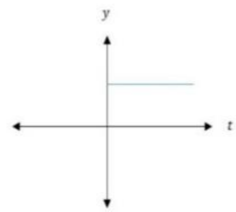
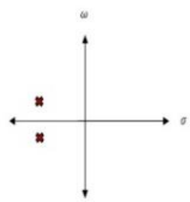
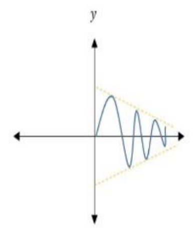
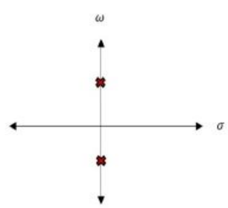
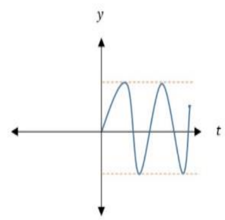
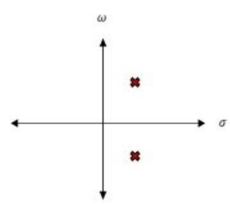
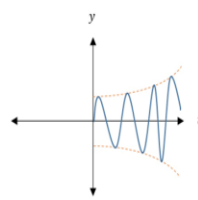
There are three types of system stability [31] we will explain each of these types (absolutely stable system, conditionally stable system and marginally stable system).

a) Absolutely stable system: the control system is absolutely stable for all ranges of system component values. The system is stable if all the poles of the present transfer function are in left half of S-plan.
b) Conditionally stable system: If the system is stable for certain ranges of system component values.
c) Marginally stable system: the system is said to be stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input [3].

The control system is marginally stable if any two poles of the transfer function are present on the imaginary axis.

Impulse responses for various root locations in the s-plan:

$Type\ of\ root$	$s - plan\ graph$ $s = \sigma + i\omega$	$Response\ graph$	$type\ stable$
Real and Negative			Absolutely stable

Real and Positive			Unstable
Zero			Marginally stable
Conjugate complex With Negative Real part			Conditionally stable
Conjugate Imaginary			Marginally stable
Conjugate With Positive real part			unstable

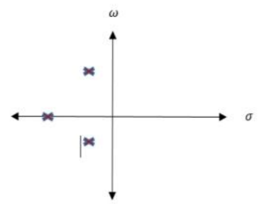
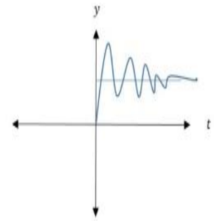
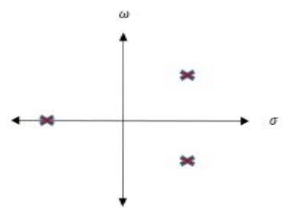
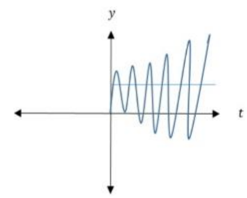
Complex With Negative Real part			Conditionally stable
Conjugate With Positive real part and one pole Negative			unstable

Table3.1

3-3 The Routh-Hurwitz Stability Criterion

The Routh stability criterion is an analytical procedure for determining if all the roots of a polynomial have negative real parts, and it is used in the stability analysis of linear time invariants systems. This stability criterion is useful in various engineering applications [39-40-41]

It is important to obtain the sign of the $G(s)$ roots not the value of them. To distinguish the sign of $D(s)$ can be use Routh stability criterion. After reading the theory of the network structure [42], we can easily say that any pole of the system located on the right side of the origin of the s- plane, makes the system unstable based on this condition [43]. E.J.Routh began to verify the necessary and sufficient conditions for the stability of the system. This criterion is also known as modified Hurwitz Criterion of stability of the system. We will study this criterion in two parts. Part one will cover necessary conditions for stability of the system and part two will cover the sufficient conditions for the stability of the system.

One-The necessary conditions: it includes two parts as follows

All the coefficients of the characteristic of polynomial (equation 3.1) should be positive and real.

I. All the coefficients of the characteristic polynomial should be non-zero.

Two-The sufficient condition:

The helpful key to identify whether a system is stable or not is to consider that all elements of the first column of the Routh table have same sign.

This method yields stability information without the need to solve closed –loop system poles.

Using this method, we can tell the number of the closed –loop system poles in the half –plan, in the right half-plan, and on the $i\omega - axis$(notice that we say how many not where).

The method requires two steps:

1-generate a data table called a Routh table:

i. Begin by labeling the rows with powers of s form the highest power of the denominator polynomial to s_0.

ii. The first row will consist of all the even terms of the characteristic equation. Arrange them from first (even term) to last (even term). The first row is written below: a_0, a_2, a_4, a_6, ...

iii. The second row will consist of all the odd terms of the characteristic equation. Arrange them from first (odd term) to last (odd term). The first row is written below: a_1, a_3, a_5, a_7, ...

iv. The elements of third row can be calculated as follows

a) First element: Multiply a_0 with the diagonally opposite element of next column (a_3) then subtract this from the product of $\mathrm{a}_1$and a_2 (where a_2 is diagonally opposite element of next column) and finally divide the result so obtain witha_1. Mathematically we write as first element

$$\mathrm{b}_1 = \frac{\mathrm{a}_1\mathrm{a}_2 - \mathrm{a}_3\mathrm{a}_0}{\mathrm{a}_1}$$

b) Second element: Multiply a_0 with the diagonally opposite element of one column after the next column (a_5) then subtract this from the product of $\mathrm{a}_1$and a_4 (where, a_4 is diagonally opposite element of next-to-next column) and then finally divide the result then obtain with a_1. Mathematically we write second element as follows:

$$\mathrm{b}_2 = \frac{\mathrm{a}_1\mathrm{a}_4 - \mathrm{a}_5\mathrm{a}_0}{\mathrm{a}_1}$$

c) Third element: Multiply a_0 with the diagonally opposite element of one column after the next column (a_7). then subtract this from the product of $a_1$and a_6 (where, a_6 is diagonally opposite element of next-to-next column) and then finally divide the result so obtain with a_1. Mathematically we write as second element

$$b_3 = \frac{a_1a_6 - a_7a_0}{a_1}$$

v. The elements of the fourth row can be calculated through using the following procedure:

a) First element: Multiply a_1 with the diagonally the opposite element of next column (b_2) then subtract this from the product of a_3 and b_1(where, $b_1$is diagonally the opposite element of next column) and then finally divide the result so obtain withb_1. Mathematically we write the first element

$$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$$

b) First element: Multiply $a_1$with the diagonally the opposite element of the next column (b_3) then subtract this from the product of $a_5$and b_3(where, $b_1$is diagonally opposite element of next column) and finally divide the result to obtain b_1. Mathematically we write the first element as follows:

$$c_2 = \frac{b_1a_5 - b_3a_1}{b_1}$$

c) Similarly, we can calculate all the elements of the fourth row.

vi. Similarly, we can calculate all the elements of all rows.

We will explain this on the table3.2:

The Characteristic equation is:

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s^1 + a_ns^0 = 0$$

s^n	a_0	a_2	a_4	a_6	...
s^{n-1}	a_1	a_3	a_5	a_7	...
s^{n-2}	$b_1 = \frac{a_1a_2 - a_3a_0}{a_1}$	$b_2 = \frac{a_1a_4 - a_5a_0}{a_1}$	$b_3 = \frac{a_1a_6 - a_7a_0}{a_1}$	...	...

s^{n-3}	$c_1 = \dfrac{b_1a_3 - b_2a_1}{b_1}$	$c_2 = \dfrac{b_1a_5 - b_3a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$			
s^0	a_n				

Table3.2

We can multiply positive constant value and for the equation if not found s to power i that means it's coefficient equal 0.

Stability criteria: if all the elements of the first column are positive then the system will be stable. However, if anyone of them is negative the system will be unstable. The number of roots of polynomial that are in the RHP is equal to the number of sign changes in first column.

Example 3.1

Determine if the following characteristic equation represents a stable system.

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Solution:

We find the Routh table:

s^4	1	3	1
s^3	3	2	0
s^2	$\frac{7}{3}$	1	0
s^1	$\frac{5}{7}$	0	0

s^0	1	0	0

The system is stable because in the first column there are not changes of sign.

Example 3.2

Apply the Routh-Hurwitz criterion to the following characteristic equation to determine the number of roots in the right half s-plane.

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Solution:

we find the Routh table:

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	-6	0	0
s^0	5	0	0

It is unstable because two poles in sign appear in the first column (In the first column there are two changes of sign, from1 to -6 and from -6 to 5); we find that two roots of the characteristic equation lie on the right-hand plane (RHP)of the s-plan.

Example 3.3

Find the stability of the continuous system having the characteristic polynomial of the third order system given below:

$$s^3 + s^2 + 2s + 24 = 0$$

Solution:

we find the Routh table:

s^3	1	2
s^2	1	24
s^1	-22	0
s^0	24	0

The third element of the first column is negative, $i.e.$ the transformation is from positive to negative and negative to positive (In the first column there are two changes of sign, from 1 to -22 and from -22 to 24) and this indicates the presence of two poles on the right-hand plane (RHP) of the s-plan.

3-3-1 Special cases in Routh Stability Criteria

There are some special cases related to Routh Stability Criteria, which are discussed below:

Case 1

It is when the first term in any row of the array is zero while the rest of the row has at least one none zero term.

In this case we will assume a very small value (ε) which is tending to zero in place of zero. By replacing zero with (ε) we will calculate all the elements of the Routh array. After calculating all the elements, we will apply the limit at each element containing (ε). Through solving the limit at every element if we get positive limiting value, we will say the given system is stable otherwise in all other conditions the given system is not stable.

Example 3.4

Check whether the given system in figure3.2 is stable or unstable by Routh criterion:

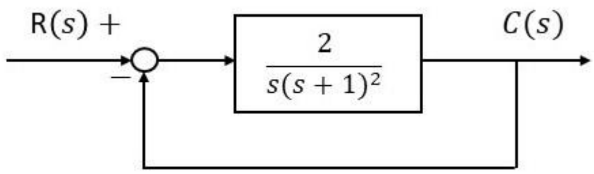


Figure3.2

Solution: The characteristics equation

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{2}{s(s+1)^2} = 0$$

$$\Rightarrow s^3 + 2s^2 + s + 2 = 0$$

Let's create a Routh table

s^3	1	1
s^2	2	2
s^1	0	0
s^0	?	

We notice that the third element of the first column is zero, so we deleted the zero and put a value very close to zeroϵ ($\epsilon \to 0$), and then we completed the Routh table.

s^3	1	1
s^2	2	2
s^1	ϵ	0
s^0	2	0

The system is stable.

Example 3.5

Determine the stability of the system having a characteristic equation given below:

$$D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Solution: Let's create a Routh table

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2	?		
s^1			
s^0			

We notice that the third element of the first column is zero, so we deleted the zero and put a value very close to zeroϵ ($\epsilon \to 0$). We can calculate the element:

$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon}$ and $d_1 = \frac{6c_1 - 10\epsilon}{c_1} \to 0$

Then we complete the Routh table.

s^5	1	2	11
s^4	2	4	10
s^3	ϵ	6	0
s^2	c_1	10	0
s^1	d_1	0	0
s^0	10	0	0

Note that c_1 is negative i.e., the transformation from positive to negative and negative to positive, the system is unstable and this indicates the presence of two poles in the right-hand plane (RHP)of the s-plan.

Example 3.6

Make the Routh table for the system as below:

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Solution: The characteristic equation for system is

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

Then we complete the Routh table

s^5	1	3	5
s^4	2	6	3
s^3	0	$\frac{7}{2}$	0
s^2	?		
s^1			
s^0			

We notice that the third element of the first column is zero, so we deleted the zero and put a value very close to zero ϵ ($\epsilon \to 0$). We can calculate the

$$c_1 = \frac{6\epsilon - 7}{\epsilon} = \frac{-12}{\epsilon} \ and \ d_1 = \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$$

s^5	1	3	5
s^4	2	6	3
s^3	ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon-7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	0	0
s^0	3	0	0

The ϵ It is a very close value to zero. It can be a negative value close to zero or a positive value close to zero. To analyze the stability of this system, we can suppose the ϵ positive and negative as on the table below:

$Label$	$first\ column$	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	ϵ	+	−
s^2	$\frac{6\epsilon-7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	+	+
s^0	3	+	+

If we assume that the value of ϵ is positive, then the fourth element of the first column is negative, and this indicates the presence of a change in the sign from positive to negative then to positive, and this indicates the presence of two of poles in the right-hand plane (RHP) of the s-plan.

Also, if we assume that the value of ϵ is negative, we notice a change in the sign in the third element from the first column, this indicates the presence of two poles in the right - hand plane (RHP) of the s-plan. This means that the system is unstable.

Case 2

When all the elements of any row of the Routh array are zero. In this case we can say the system has the symptoms of marginal stability. Let us first understand the physical meaning of having all the elements zero of any row. The physical meaning is that there are symmetrically located roots of the characteristic equation in the s plane. Now in order to find out the stability in this case we will first find out auxiliary equation. Auxiliary equation can be formed by using the elements of the row just above the row of zeros in the Routh array. After finding the auxiliary equation, we will differentiate the auxiliary equation to obtain elements of the zero row. If there is no sign change in the new Routh array formed by using auxiliary equation, then the given system is a limited stable. However, in all other cases we will say, the given system is unstable.

Example 3.7

Make the Routh table for the system shown and determine the pole distribution in the $s - plan$ the auxiliary polynomial method

$$G(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Solution: The characteristic equation for system is

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$

Then we complete the Routh table:

s^5	1	6	8
s^4	1	6	8
s^3	0	0	0
s^2	?		

s^1			
s^0			

Here to complete Routh table, we derivative the equation for the second row

$$\frac{d(s^4+6s^2+8)}{ds} = 4s^3 + 12s \qquad (3)$$

We substitute the coefficients of the equation (3) in the third row instead of zero, and divide the third row by 4 becomes the table below:

s^5	1	6	8
s^4	1	6	8
s^3	1	3	0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

We notice that the system is stable because the first column did not change its signal.

Example 3.8

Determine a range of values of system parameter p for which the system is stable.

$$s^3 + 3s^2 + 3s + 1 + p = 0$$

Solution: Let's create a Routh table

s^3	1	3	0

s^2	3	$1+p$	0
s^1	$\frac{8-p}{3}$	0	0
s^0	$1+p$	0	0

Note that p appears twice in the first column in the third element and the fourth through $\frac{8-p}{3}$ must be $8-p>0$ in the fourth element $1+p$ must be $1+p>0$

Thus, for the system to be stable, it must be $-1<\mathrm{p}<8$.

Example 3.9

Determine the range of parameter p for which the system is unstable.

$$D(s)=s^4+s^3+s^2+s+p$$

Solution:

Let's create a Routh table

s^4	1	1	p
s^3	1	6	0
s^2	ϵ	p	0
s^1	$\frac{\epsilon-p}{\epsilon}$	0	0
s^0	p	0	0

Note that $c_1=\frac{\epsilon-p}{\epsilon}\to\frac{-p}{\epsilon}$ Thus, the value of the p must be $p<0$. In the fourth element of the first column, we note that the value of the p must be $p>0$, and this leads to a contradiction, so the system is unstable.

3-4 Lyapunov Stability

Stability is one of the most important properties characterizing a system's qualitative behavior. There are a number of stability concepts used in the study of dynamical systems. But perhaps the most important stability concept is that of stability in the sense of Lyapunov or simply Lyapunov stability. This concept is not a property of a system, it is a property of the system's equilibrium point which says that we can make the future states of system remain arbitrarily close to the equilibrium by simply taking the initial condition close enough. In this regard, it is ultimately a statement about the continuity of the flows in the neighborhood of the equilibrium point.

From an application's standpoint, this notion of remaining "close" to an equilibrium lies at the heart of the regulation problem. A control system is designed to regulate its response in the neighborhood of an operating point. In other words, whether or not that operating point can be maintained under small input disturbances is ultimately related to Lyapunov stability in that the system cannot diverge from that neighborhood for sufficiently small disturbances.

The Lyapunov method is suitable for linear and nonlinear systems of any order. For linear systems, it provides sufficient and necessary conditions. For nonlinear systems it only provides sufficient conditions for asymptotic stability. It is shown that the stability of the system can be maintained without solving and determining the state equation. The roots of the characteristic equation of the system. The method depends on the definition The Lyapunov function is closely related to the energy function of the system. for linear systems, such functions are easy to determine; unfortunately for nonlinear This can be very difficult for the system. We only consider linear systems, although these are simple examples for illustration the problem of use can be solved by other basic methods.

As a very simple illustrative example of a trajectory, we consider the motion of an undamped pendulum (see Figure 3.3.). The equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0 \qquad (3.3)$$

which is a nonlinear equation.

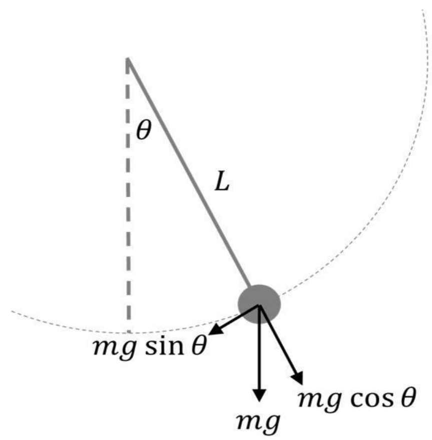


Figure 3.3

Although we were able to find trajectories in the phase plane of this very simple nonlinear example, in this dissertation we focus primarily on linear systems. On the other hand, assuming the displacement is small, we have

$$\sin(\theta) \cong \theta$$

And equation (3.3) then becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \qquad (3.4)$$

We again use the "linearization technique" first mentioned in Chapter one (1-2). Note that a similar linear equation is obtained for the motion of a mass (m) on a spring of stiffness (k); the equation is

$$\ddot{x} + \frac{k}{m}x = 0$$

This is a well-known linear equation of simple harmonic motion, and has the

Solution

$$\theta = Asin(\sqrt{\frac{g}{Lt}} + \epsilon\ ) \qquad (3.5)$$

where A and ϵ are constants determined by the initial conditions. To write equation (3.4) in state-space form we define the variables

$$x_1 = \theta \ \text{ and } \ x_2 = \dot{\theta}$$

So that

$$\dot{x}_1 = x_2 \ \text{ and } \ \dot{x}_2 = -\frac{g}{Lx_1} \qquad (3.6)$$

Now we find the solution to equation (3.6), in this case (using equation (3.5))

$$x_1 = Asin\left(\sqrt{(\frac{g}{L})}t + \epsilon\right)$$

$$x_2 = A\sqrt{(\frac{g}{L})}\cos\left(\sqrt{(\frac{g}{L})}t + \epsilon\right) \qquad (3.7)$$

And assume in trajectory by eliminating t in equation (3.6), it is

$$x_1^2 + \frac{x_2^2}{\sqrt{(\frac{g}{L})}} = A^2 \qquad (3.8)$$

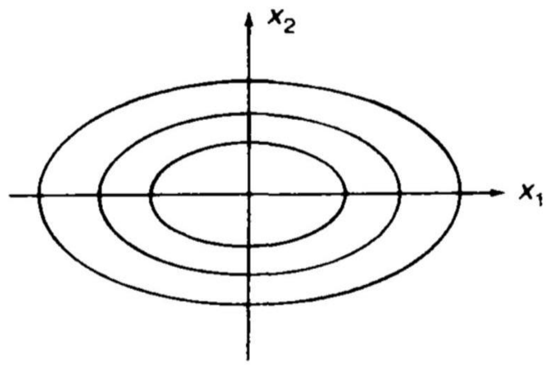


Figure3.3: The trajectories (for various values of A and e) are ellipses.

Each trajectory shown in Figure 3.3 is closed, indicating that the pendulum would swing infinitely if there were no resistance to its motion. Closed trajectories are typical of periodic motion, i.e., the solution $x(t)$ has the following properties

$$x(t + T) = x(t)$$

Where $T$is the period of the motion.

If we assume a more realistic situation where the oscillations are suppressed, the trajectory will end at $x = 0$, so it may have the shape shown in Figure 3.4. (Arrows indicate direction of movement as t increases.)

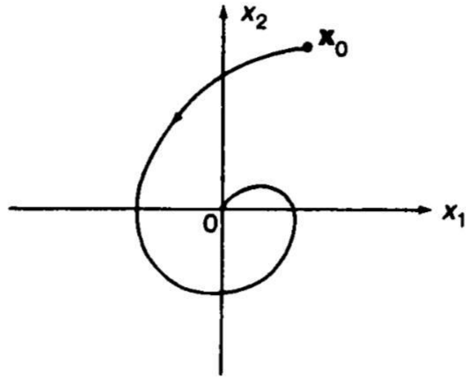


Figure3.4

In general, autonomous systems, the state equation takes the form of equation (3.6).

$$\dot{x}_1 = P(x_1, x_2) \text{ and } \dot{x}_2 = Q(x_1, x_2) \qquad (3.8)$$

The solution is

$$x_1 = \phi(t) \text{ and } x_2 = \psi(t) \qquad (3.9)$$

And can be plotted to give the trajectory.

For a linear system, there is only one equilibrium point (we assume the matrix A is not singular), which is the point $x = 0$. This is clearly the case for the linearized pendulum equation (3.6); the equilibrium point is $(\theta = 0, \dot{\theta} = 0)$. When the system is perturbed from the equilibrium position $x = 0$, the trajectory of the state vector x determines the stability or instability of the equilibrium point, thus determining system (in the case of a linear system.) Under the given initial conditions, the trajectory is completely determined by the system's equation of state.

In fact, we can see from the discussion of the pendulum problem that the equilibrium point $x = 0$ appears to be stable, since a trajectory starting "closer" to that point will always be at a "reasonable" distance from it. On the other hand, if the system is stable and the trajectory eventually approaches $x = 0$ at $t \to \infty$, then the system is asymptotically stable (i.e. damped oscillations). In mathematical terms, these definitions take the form shown in

Figure3.5:

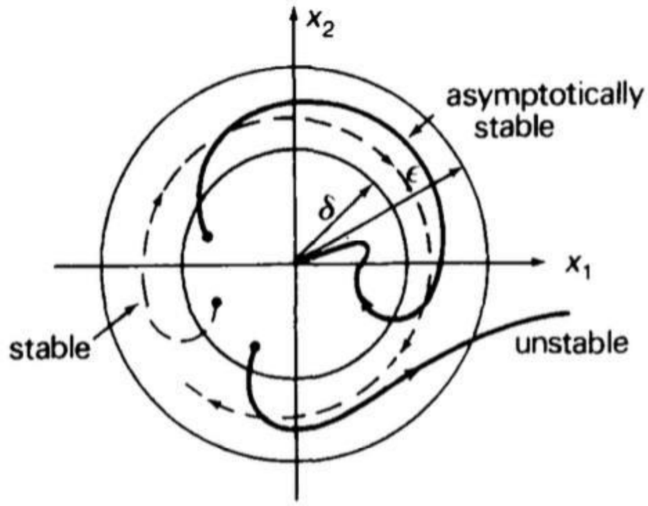


Figure3.5

3-4-1 Stability in the sense of Lyapunov

The equilibrium point $x = 0$ of is stable (in the sense of Lyapunov) at $t = t_0$ $(t > 0)$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that

$$\| x(t_0) \| < \delta \implies \| x(t) \| < \varepsilon, \qquad \forall t \geq t_0 \qquad (3.10)$$

Lyapunov stability is a very mild requirement on equilibrium points. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. Also, stability is defined at a time instant t_0. Uniform stability is a concept which guarantees that the equilibrium point is not losing stability. We insist that for a uniformly stable equilibrium point x, δ in the Definition (3-3-1) not be a function of t_0, so that equation (3.10) may hold for all t_0.

3-4-2 Asymptotic stability

An equilibrium point $x = 0$ is asymptotically stable at $t = t_0$ if $x = 0$ is stable, and

$x = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\| x(t_0) \| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0. \qquad (3.11)$$

As in the previous definition, asymptotic stability is defined at t 0. Uniform asymptotic stability requires:

1. $x = 0$ is uniformly stable, and

2. $x = 0$ is uniformly locally attractive; i.e., there exists δ indepen- dent of t_0 for which Equation (3.11) applies. In addition, the convergence of formula (3.11) is required to be consistent.

3-4-3 Unstable

Finally, if the equilibrium point is unstable, we say it is unstable. This is not as tautological as it sounds, and the reader should be assured that he or she can negate Lyapunov's definition of stability in order to obtain a definition of instability. In robotics we are almost always interested in uniform asymptotically stable equilibria. When we want to move the robot to a point, we actually want to converge to that point, not just stay nearby. Figure 3.6 illustrates the difference between stability in the Lyapunov sense and asymptotic stability.

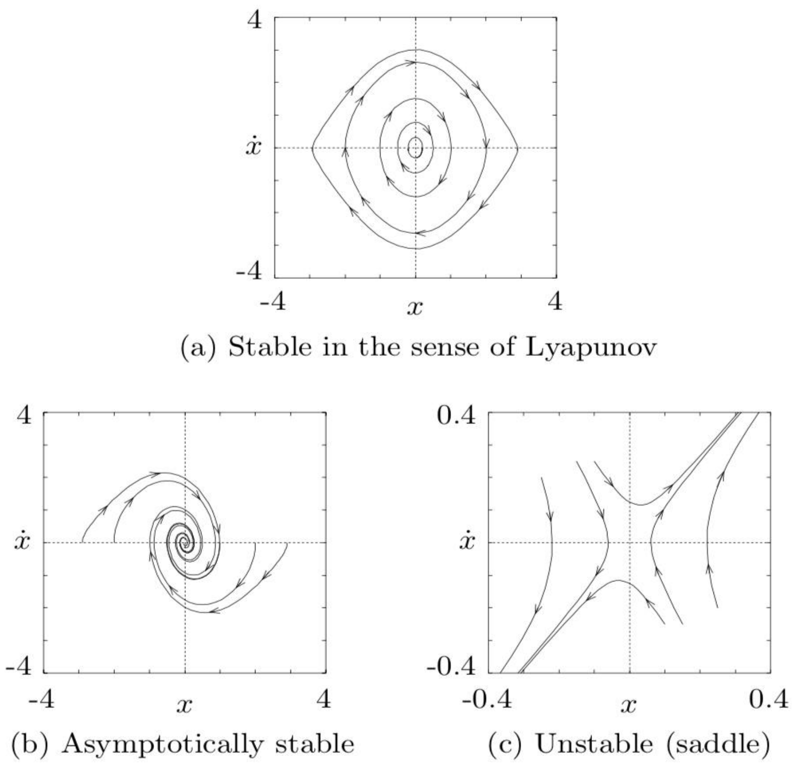


Figure 3.6: Phase portraits for stable and unstable equilibrium points.

In the definition, the starting point "closer" to the equilibrium point is defined by the "$\delta -$ neighborhood", and the "reasonable" distance is the "$\epsilon -$neighborhood". Let us consider an unconstrained system with an equation of state

$$\dot{x} = Ax(t) \qquad (3.12)$$

We have found the solution

$$x = e^{At}x(0) \qquad [e^{At} = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + A_3e^{\lambda_3 t} + \cdots + A_ne^{\lambda_n t}]$$

$$x = \left(e^{At} = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + A_3e^{\lambda_3 t} + \cdots + A_ne^{\lambda_n t}\right)x(0) \qquad (3.13)$$

For asymptotic stability

$$\lim_{t\to\infty}\|x(t)\| = 0$$

Where $\|x\|$is norm of the vector defend by

$$\|x\|^2 = (x, x) = x_1^2 + x_2^2 + \cdots + x_n^2$$

Therefore, for asymptotic stability, all state variables $x_1, x_2, \ldots, x_n$must decrease to zero when $t \to \infty$. From equation 3.13 we see that this is the case when

$Re\lambda_i < 0$ $(i = 1,2, \ldots, n)$, where λ_i is the eigenvalue of A, i.e., the solution of the characteristic equation

$$|sI - A| = 0$$

So we have the same problem as in the previous section, only now We examined the system equation of state in the previous section Consider the system transfer function. Back to Lyapunov his approach relies on state railways , but not directly. He suggested that there might be trajectory-related features whose properties might determine whether a trajectory ends at the origin.

3-4-4 Lyapunov function

Definition 3.1

a) A scalar function $V(x_1, x_2, \ldots, x_3)is$ said to be positive definite if $V(x)$is such that $V(x)$ is positive at all points in the state space except at origin $x = 0$ where $V(x)$ equal to zero.
   i.e. Function $V(x): U \subseteq R^2 \to R$ is said be positive definite in U if
   (i) $V(0) = 0$ and
   (ii) $V(x) > 0$ for all $x \in U$ such that $x \neq 0$).

b) A scalar function $V(x_1, x_2, \ldots, x_3)is$ said to be negative definite if $V(x)$is such that $V(x)$ is negative at all points in the state space except at origin $x = 0$ where $V(x)$equal to zero.
   I.e. a continues function $V(x)$ is negative definite if $V(x) \geq 0$ for all $x \in U$ and $V(x)$ is said be negative definite if

(i)$V(0) = 0$ and
(ii) $V(x) < 0$ for all $x \in U$ such that $x \neq 0$.

c) A scalar function $V(x_1, x_2, \dots, x_3) is$ said to be positive-semi-definite if $V(x)$is such that $V(x)$ is positive at all points except at one or more points in the state space including the origin $x = 0$ where $V(x)$equal to zero.
I.e. a continues function $V(x)$ is positive-semi-definite if $V(x) \geq 0$ for all $x \in U$.

d) A scalar function $V(x_1, x_2, \dots, x_3) is$ said to be negative-semi-definite if $V(x)$is such that $V(x)$ is negative at all points except at one or more points in the state space including the origin $x = 0$ where $V(x)$equal to zero.
i.e., a continues function $V(x)$ is negative-semi-definite if $V(x) \leq 0$ for all $x \in U$.

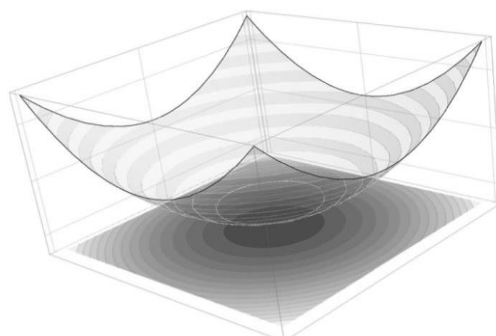


Figure 3.7: Graph of a positive definite function $V(x)$ and its level sets.

$V(x)$ is called Lyapunov function if it has the following properties:

a) $V(x)$ and its partial derivatives $\frac{\partial V}{\partial x_i}$ $(i = 1,2,\dots,n)$ are continuous.
b) $V(x) > 0$ for $x \neq 0$ (in any neighborhood of the origin) and $V(0) = 0$.
c) $\dot{V}(x) < 0$ for $x \neq 0$ (in any neighborhood of the origin) and $\dot{V}(0) = 0$.

Example 3.10

Classify the following functions according to the above definitions:

1) $V(x) = x_1^2 + x_2^2 \quad , x \in R^2$
2) $V(x) = x_1^2 \quad , x \in R^2$
3) $V(x) = -(x_1 + x_2)^2 \quad , x \in R^2$
4) $V(x) = x_1^2 + 2x_2^2 - 4x_1x_2 \quad , x \in R^2$

Solution:

1) $V(0) = 0$

$V(x) > 0 for\ all\ x \neq 0$
$\Longrightarrow V(x)$ is positive-definite.

2) $V(x) \geq 0$
There are points other than 0at which $V(x) = 0$
For example, $x = (0, c)$ where $c \in R$is any nonzero number.
$\Longrightarrow V(x)$ is positive-semi-definite

3) $V(x) \leq 0$
For $x_1 = -x_2$,$V(0) = 0$
For example, $x = (0, c)$ where $c \in R$is any nonzero number.
$\Longrightarrow V(x)$ is negative-semi-definite.

4) $V(1,1) < 0, V(1,3) > 0$
$\Longrightarrow V(x)$ is indefinite.

3-4-5 Lyapunov stability theory

The origin (that is, the equilibrium point) is asymptotically stable if there exists a Lyapunov function in the neighborhood of the origin.

If $\dot{V}(x)$ is only negative-semi-definite, that is, $\dot{V}(x) \leq 0$, then the origin is a stable pint.

Proof

We consider $\dot{V}(x) \leq 0$ then $V(x) = \mathrm{K}$ ($\mathrm{K}\ is\ a\ constant$) and this represents a closed curve enclosing the origin.

By using chain rule

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]\left[\frac{dx_1}{dt}, \frac{dx_2}{dt}\right], \qquad [also\ \dot{V}(x) = \dot{V}(x_1, x_2)]$$

And we can write x_1, x_2 the vector

$$\left[\frac{dx_1}{dt}, \frac{dx_2}{dt}\right]^T = \frac{dx}{dt}$$

Also, is the tangent vector to the trajectory. On the other hand

$$\left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right] = grad\ V$$

is a vector normal to the surface $V(x_1, x_2) = \mathrm{K}$

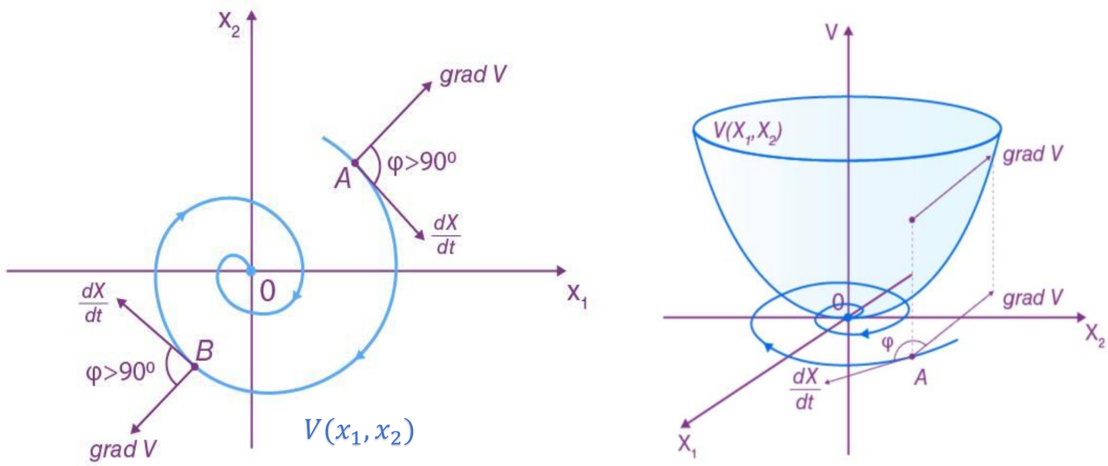


Figure 3.8

Hence

$$\dot{V}(x_1, x_2) = \alpha B cos\varphi \leq 0$$

where α and B are the magnitudes of the two vectors involved, $\frac{dx}{dt}$ and $grad\ V$, and φ is the angle between them.

It follows that $\frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2}$.

If $\dot{V}(x) = 0$ then $\varphi = \frac{\pi}{2}$ and the direction of the motion on the trajectory is tangent (that is, along) the curve.

If $\dot{V}(x) < 0$, then $\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$, that is, the direction of the motion on the trajectory is into the curve. Notice that no trajectory exists on which the motion would be outwards from the curve.

It follows that once the trajectory starts inside the curve $V(x) = K$, it never moves outward across the curve. This proves that the origin is stable according to the criteria discussed earlier in this section. This is sometimes called stability in the Lyapunov sense. The asymptotic stability of the origin follows similar reasoning when $\dot{V}(x)$ is negative-definite.

3-4-6 Quadratic forms

Definition 3.2

A quadratic form is a scalar function $V(x)$ of variables $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ defined by

$$V(x) = x^T P x \,, \qquad P = P^T$$

is a positive-definite function, and

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

We know that $\dot{x} = Ax$

$$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

$$= x^t (A^T P + PA) x \qquad (3.14)$$

i.e. the system is asymptotically stable if the following condition is satisfied

$$A^T P + PA < 0$$

or, equivalently

$$A^T P + PA = -Q, \quad Q = Q^T > 0 \qquad (3.15)$$

Where Q is the positive definite matrix. Remember that all eigenvalues of a positive definite matrix lie in the closed right half of the complex plane. The algebraic matrix equation (3.15) is called the algebraic Lyapunov equation. For more information on this important equation and its role in system stability and control, see Gajic and Qureshi (1995). We are now able to formulate a Lyapunov stability theory for linear continuous time-invariant systems.

Theorem 3.1

The linear time invariant system $\dot{x} = Ax$ is asymptotically stable if and only if for any $Q = Q^T > 0$ there exists a unique $P = P^T > 0$ such that (3.15) is satisfied.

Theorem 3.2

The time invariant linear system $\dot{x} = Ax$ is asymptotically stable if and only if the pair (A, C) is observable and the algebraic Lyapunov equation (3.15) has a unique positive definite solution.

Example 3.11

Consider the same system matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$ with the matrix Q obtained from

$$Q_1 = C^T C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [0 \quad 0 \quad 1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the pair (A, C) is observable since $rank\{(A, C)\} = 3$. The algebraic Lyapunov equation

$$A^T P_1 + P_1 A = Q_1$$

has the positive definite solution

$$P_1 = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix} > 0$$

which can be confirmed by finding the eigenvalues of P_1, so that the considered linear system is asymptotically stable.

3-4-7 Direct method of Lyapunov

Let the system is given by $\dot{x} = Ax$ select the Lyapunov function as $V(x) = x^T P x$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P (Ax)$$

$$= x^t (A^T P + PA) x$$

$\dot{V}(x) = x^T Q x$ When $Q = A^T P + PA$

For system to be astable $\dot{V}(x)$ is positive definite

$\dot{V}(x) = x^T (-Q) x$ where $-Q = A^T P + PA$

If P is positive definite the system is stable.

Example 3.12

Determine the stability of system $\dot{x} = Ax$

where

$$\dot{x}_1 = -x_1 - 2x_2$$
$$\dot{x}_2 = x_1 - 4x_2$$

And Q = I.

Solution:

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

and from equation(3.15) become

$$-Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A^T = \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix}$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -p_{11} + p_{21} & -p_{12} + p_{22} \\ -2p_{11} - 4p_{21} & -2p_{22} - 4p_{22} \end{bmatrix} + \begin{bmatrix} -p_{11} + p_{12} & -2p_{11} - 4p_{12} \\ -p_{21} + p_{22} & -2p_{22} - 4p_{22} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2p_{11} + 2p_{21} & -2p_{11} - 5p_{12} + p_{22} \\ -2p_1 - 5p_{12} + p_{22} & -2p_{12} - 8p_{22} \end{bmatrix}, [p_{12} = p_{21}]$$

$-2p_{11} + 2p_{21} = -1$ (1)

$-2p_{11} - 5p_{12} + p_{22} = 0$ (2)

$-2p_1 - 5p_{12} + p_{22} = 0$ (3)

$-2p_{12} - 8p_{22} = -1$ (4)

By solving equation (1,2,3and4) we get

$$p_{11} = \frac{23}{60}, p_{12} = -\frac{7}{60}, p_{21} = -\frac{7}{60}, p_{22} = \frac{11}{60}$$

$$P = \begin{bmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{bmatrix}$$

$$P_1 = \left|\frac{23}{60}\right| = \frac{23}{60} > 0$$

$$P_2 = \begin{vmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{vmatrix} = 204 > 0$$

Hence P is positive definite, so system is asymptotically stable.

Example 3.13

Use Lyapunov's direct method to determine $\dot{x} = Ax$ when $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and Q = I write the Lyapunov function, are the system asymptotically stable?

Solution:

$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and from equation(3.15) become

$$-Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A^T = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\text{And } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - p_{21} & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} -p_{12} & p_{11} - p_{12} \\ -p_{22} & p_{21} - p_{22} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2p_{21} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2p_{12} - 2p_{22} \end{bmatrix}, [p_{12} = p_{21}]$$

$-2p_{21} = -1$ (1)

$p_{11} - p_{12} - p_{22} = 0$ (2)

$p_{11} - p_{12} - p_{22} = 0$ (3)

$2p_{12} - 2p_{22} = -1$ (4)

By solving equation (1,2,3and4) we get

$$p_{11} = \frac{3}{2}, p_{12} = \frac{1}{2}, p_{21} = \frac{1}{2}, p_{22} = 1$$

$$P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$P_1 = \left|\frac{3}{2}\right| = \frac{3}{2} > 0$$

$$P_2 = \begin{vmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{vmatrix} = \frac{5}{4} > 0$$

Hence P is positive definite, so system is asymptotically stable.

$$V(x) = x^T P x$$

$$V(x) = [x_1 \quad x_2] \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$V(x) = \frac{3}{2} x_1^2 + x_2^2 + x_1 x_2$$

3-5 Lyapunov Routh stability method

Sometimes it is difficult to check the stability of a problem by Lyapunov's method, and in this case, we can use Routh method for help. We call this method Lyapunov Routh method (composition of Lyapunov and Routh). In this method, first we use Lyapunov's method to obtain the form of matrix in state space:

$$\dot{X} = AX + BU$$

Now we use the Routh method to write a specific sentence of the matrix A and check the stability of the system. As shown in the examples (3.8 and 3.9), we can easily check the stability of a parametric system.

Example 3.14

Determine the range of K for the system in figure 3.9 to be asymptotically stable

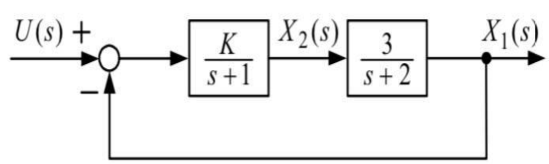


Figure 3.9

Solution: using Lyapunov's method,

$$X_1 = \frac{3}{s+2} X_2 \quad \Rightarrow \; \dot{x_1} = -2x_1 + 3x_2$$

$$X_2 = \frac{K}{s+1}(-X_1 + U) \quad \Rightarrow \; \dot{x_2} = -kx_1 - x_2 2ku$$

The state equation is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -k & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} u$$

$$A = \begin{bmatrix} -2 & 3 \\ -k & -1 \end{bmatrix} \ and\ B = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

Solving $A^T P + PA = -I$

$$\begin{bmatrix} -2 & -k \\ 3 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -k & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We obtain

$$p = \frac{1}{18k + 12} \begin{bmatrix} k^2 + 3k + 3 & 3 - 2k \\ 3 - 2k & 3k + 15 \end{bmatrix}$$

Here it is difficult to calculate the value k

So, we use a simpler method to known

The stability of system by using Routh stability criterion:

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -2 & 3 \\ -k & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{bmatrix} -2 - \lambda & 3 \\ -k & -1 - \lambda \end{bmatrix}$$

$$= 2 + 2\lambda + \lambda + \lambda^2 + 3k = \lambda^2 + 3\lambda + 2 + 3k$$

By using Routh table:

λ^2	1	$2 + 3k$
λ^1	3	0
λ^0	$2 + 3k$	0

The system is stable

$$2 + 3k > 0 \Rightarrow k > -\frac{2}{3}$$

Example 3.15

Determine a range of values of system parameter K for which the system in figure 3.10

is stable.

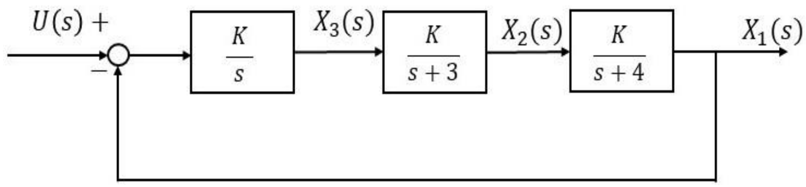


Figure 3.10

Solution: using Lyapunov's method

$$X_1 = \frac{3}{s+4}X_2 \quad \Longrightarrow \; \dot{x_1} = -4x_1 + Kx_2$$

$$X_2 = \frac{K}{s+3}X_3 \quad \Longrightarrow \; \dot{x_2} = -3x_2 + Kx_3$$

$$X_3 = \frac{K}{s}(-X_1 + U) \Longrightarrow \dot{x_3} = -Kx_1 + Ku$$

The state equation is:

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\end{bmatrix} = \begin{bmatrix}-4 & K & 0\\ 0 & -3 & K\\ -K & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ K\end{bmatrix}u$$

$$A = \begin{bmatrix}-4 & K & 0\\ 0 & -3 & K\\ -K & 0 & 0\end{bmatrix} \; and \; B = \begin{bmatrix}0\\ 0\\ K\end{bmatrix}$$

Solving $A^TP + PA = -I$

$$\begin{bmatrix}-4 & 0 & -K\\ K & -3 & 0\\ 0 & K & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix}\begin{bmatrix}-4 & K & 0\\ 0 & -3 & K\\ -K & 0 & 0\end{bmatrix}$$
$$= \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

Here it is difficult to calculate the value K.

So, we use a simpler method to known

The stability of system by using Routh stability criterion:

$|A - \lambda I| = 0$

$$\left|\begin{bmatrix} -4 & K & 0 \\ 0 & -3 & K \\ -K & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} -4-\lambda & K & 0 \\ 0 & -3-\lambda & K \\ -K & 0 & -\lambda \end{bmatrix}\right| = 0$$

$$\lambda^3 + 7\lambda^2 + 12\lambda + K^3 = 0$$

By using Routh table:

λ^3	1	12
λ^2	7	K^3
λ^1	$\frac{84 - K^3}{7}$	0
λ^0	K^3	0

Note that K appears twice in the first column in the third element and the fourth through $\frac{84-K^3}{7}$ must be $84 - K^3 > 0 \Rightarrow K < 4.38$ in the fourth element K^3 must be $K > 0$ Thus, for the system to be stable, it must be $0 < \mathrm{K} < 4.38$.

3-6 The Nyquist stability

A stability tests of time-invariant linear systems can also be derived in the frequency domain. It is called the Nyquist stability criterion. It is based on the results of a complex analysis known as Cauchy's principle of reasoning. Please note that the system transfer function is a complex function. By applying Cauchy's inference principle to the transfer function of an open-loop system, we obtain stability information of the transfer function of a closed-loop system and derive the Nyquist stability criterion (Nyquist, 1932). The importance of Nyquist stability is that it also determines the relative degree of system stability by determining the so-called phase and gain stability margins. Frequency domain controller design techniques require these stability margins. We only introduce the essence of the Nyquist stability criterion and define the phase and gain stability margins. The Nyquist method is used to study the stability of linear systems with pure time delays. For SISO feedback system the closed-loop transfer function is given by

$$M(s) = \frac{G(s)}{1+H(s)G(s)} \qquad (3.16)$$

when $G(s)$ represents the system and $H(s)$ is the feedback element. Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation

$$1 + H(s)G(s) = 0 = \Delta(s) \qquad (3.17)$$

which, in fact, represents the system characteristic equation.

we can consider the complex function

$$D(s) = 1 + H(s)G(s)$$

Its zeros are the closed poles of the transfer function. Furthermore, it is easy to see that the poles of D(s) are the zeros of M(s). At the same time, the poles of D(s) are the poles of the control system because they are contributed by the poles of H(s)G(s), which can be regarded as the transfer function of the control system - when the feedback loop is opened at a certain moment obtained when. The Nyquist stability test is obtained by applying the principles of Cauchy's argument to the complex function D(s). First, we explain the principles of Cauchy's argument.

1-6-1 Cauchy's Principle of Argument

Let $F(s)$ be an analytic function in a closed region of the complex plane given in figure 3.11 except at a finite number of points. It is also assumed that $F(s)$ is analytic at every point on the contour. Then, as travels around the contour in the - plane in the clockwise direction, the function $F(s)$ encircles the origin in the $(Re\{F(s)\}, Im\{F(s)\})$-plane in the same direction N times (see Figure 3.11), with N given by

$$N = Z - P \qquad (3.18)$$

where Z and P stand for the number of zeros and poles (including their multiplicities) of the function $F(s)$ inside the contour.

The equation (3.18) can be also written as

$$\arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

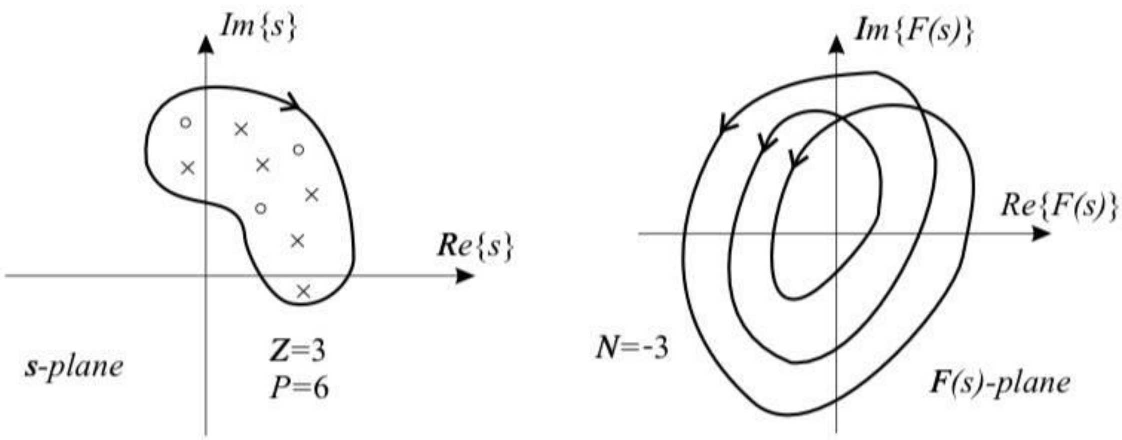


Figure 3.11: Cauchy's principle of argument

3-6-2 Nyquist plot

The Nyquist plot is a polar plot of the function $D(s) = 1 + H(s)G(s)$ when s travels around the contour given in Figure 3.12.

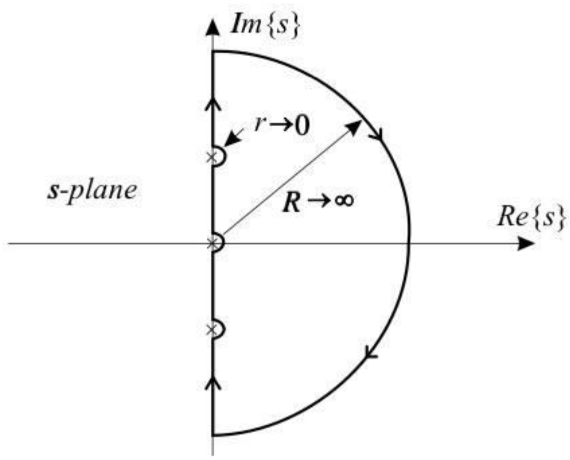


Figure 3.12: Contour in the -plane

The contour in this figure covers the whole unstable half plane of the complex plane s , $R \to \infty$. Since the function $D(s)$, according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles of $D(s)$ on the imaginary axis must be encircled by infinitesimally small semicircles.

3-6-3 Nyquist Stability Criterion

It states that the number of unstable poles in a closed loop is equal to the number of unstable poles in an open loop plus the number of orbits at the origin of the Nyquist diagram of the complex function D(s). This can be easily proven by applying Cauchy's principle of inference to a function D(s) with - plane profile shown in Figure 3.12. Note that Z and P represent the numbers of zeros and poles, respectively, of $D(s)$ in the unstable part of the complex plane. At the same time, the zeros of $D(s)$ are the closed-loop system poles, and the poles of $D(s)$ are the open-loop system poles (closed-loop zeros).

The above criterion can be slightly simplified if instead of plotting the function $D(s) = 1 + H(s)G(s)$, we plot only the function $H(s)G(s)$ and count encirclement of the Nyquist plot of $H(s)G(s)$ around the point $(-1, j0)$, so that the modified Nyquist criterion has the following form. The number of unstable closed-loop poles (Z) is equal to the number of unstable open-loop poles (P) plus the number of encirclements (N) of the point $(-1, j0)$ of the Nyquist plot of $H(s)G(s)$, that is $Z = P + N$.

3-6-4 Phase and Gain Stability Margins

Two important concepts can be derived from the Nyquist plot: phase and gain stability margin. Phase and gain stability margins are shown in Figure 3.13.

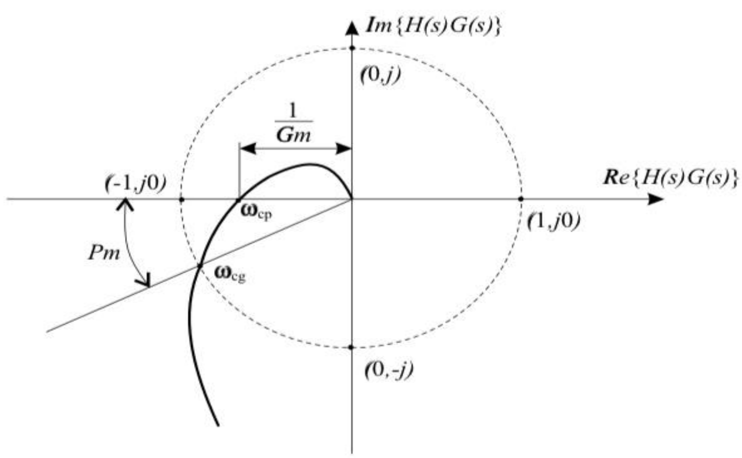


Figure 3.13: Phase and gain stability margins

They indicate the degree of relative stability; in other words: they indicate how far a given system is from the unstable region. Their formal definition is given by

$$P_m = 180^\circ + \arg\{G(jw_{cg})H(jw_{cg})\} \qquad (3.19)$$

$$G_m[dB] = 20log\frac{1}{|G(jw_{cp})H(jw_{cp})|} \qquad (3.20)$$

Where $w_{cg}$and w_{cp}stand for ,respectively, the gain and phase crossover frequencies, which form Figure 3.13 are obtained as

$$|G(jw_{cg})H(jw_{cg})| = 1 \Rightarrow w_{cg}$$

And

$$\arg\{G(jw_{cp})H(jw_{cp})\} = 180^\circ \Rightarrow w_{cp}$$

Example 3.16

Consider a control system represented by

$$G(s)H(s) = \frac{1}{s(s+1)}$$

Solution:

Since this system has a pole at the origin, the contour in the s-plane should encircle it with a semicircle of an infinitesimally small radius. This contour has three parts (a), (b), and (c). Mappings for each of them are considered below.

(a) On this semicircle the complex variable is represented in the polar form by $s = Re^{j\Psi}$with $R \to \infty, -\frac{\pi}{2} \le \Psi \le \frac{\pi}{2}$. Substituting $s = Re^{j\Psi}$ into $G(s)H(s)$, we easily see that $G(s)H(s) \to 0$. Thus, the huge semicircle from the s-plane maps into the origin in the $G(s)H(s)$-plan (see Figure 3.14)

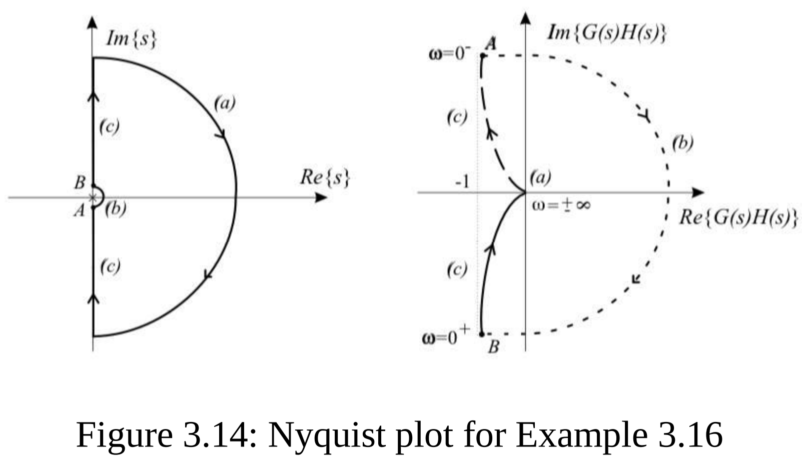


Figure 3.14: Nyquist plot for Example 3.16

(b) On this semicircle the complex variable s is represented in the polar form by $s = re^{j\Phi}$ with $r \to 0$, $-\frac{\pi}{2} \leq \Phi \leq \frac{\pi}{2}$ so that we have

$$G(s)H(s) \to \frac{1}{re^{j\Phi}} \to \infty \times \arg(-\Phi)$$

$since$ Φchange form $-\frac{\pi}{2}$at point A to $\frac{\pi}{2}$at point B, arg $\{G(s)H(s)\}$

will change from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. We conclude that the infinitesimally small semicircle at the origin in the -plane is mapped into a semicircle of infinite radius in the $G(s)H(s)$-plane.

On this part of the contour s takes pure imaginary values, i.e. $s = j\omega$ with ω changing from $-\infty$ to ∞ . Due to symmetry, it is sufficient to study only mapping along $0 \leq \omega \leq +\infty$. We can find the real and imaginary parts of the function $G(j\omega)H(j\omega)$, which are given by

$$Re\{G(j\omega)H(j\omega)\} = -\frac{1}{\omega^2 + 1}$$

$$Im\{G(j\omega)H(j\omega)\} = -\frac{1}{\omega(\omega^2 + 1)}$$

From these expressions we can see that neither the real nor the imaginary part can be set to zero, so the Nyquist plot has no intersection with the axes. For $\omega = 0^+$ we are at point B, and since the diagram ends at the origin $\omega + \infty$, the Nyquist diagram corresponding to part (c) has the shape shown in Figure 3.16. Note that the vertical asymptotes of the Nyquist diagram in Figure 3.16 are given by $Re\{G(j0^{\pm}H(j0^{\pm})\} = -1$ because at these points $Im\{G(j0^{\pm}H(j0^{\pm})\} = \pm\infty$. From the Nyquist diagram we see $N = 0$ and because there are no open poles in the left half of the complex plane, i.e.,$P = 0$, we have $Z = 0$ so that the corresponding closed-loop system has no unstable poles.

The Nyquist plot is drawn by using the MATLAB function Nyquist

$num = 1; den = [1 \quad 1 \quad 0]$

$nyquist(num, den);$

$axis([-1.5 \quad 0.5 \quad -10 \quad 10]);$

$axis([-1.2 \quad 0.2 \quad 1 \quad 1]);$

The MATLAB Nyquist plot is presented in Figure 3.17. It can be seen from Figures 3.15 and 3.16 that $\frac{1}{G_m} = 0$, which implies that $G_m = \infty$. Also, from the same figures it follows that $w_{cp} = \infty$. In order to find the phase margin and the corresponding gain crossover frequency we use the MATLAB function margin as follows

$[Gm, Pm, wcp, wcg] = margin(num, den)$

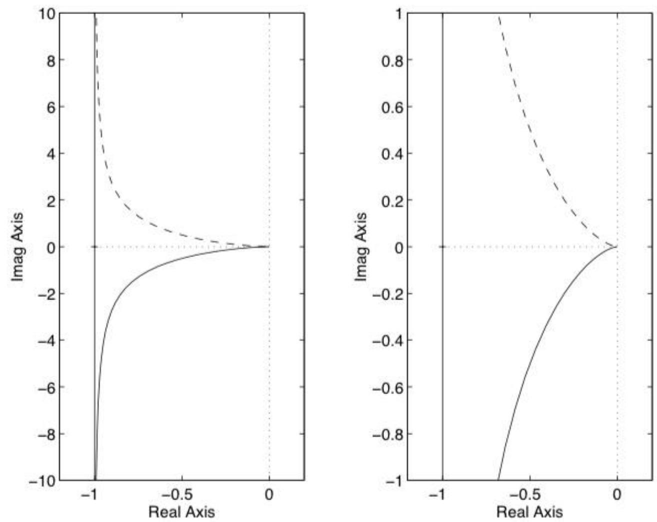


Figure 3.17: MATLAB Nyquist plot for Example 3.16

producing, respectively, gain margin, phase margin, phase crossover frequency, and gain crossover frequency. The required phase margin and gain crossover frequency are obtained as $Pm = 53.4108^{\circ}, w_{cg} = 0.7862\ rad/s$.

Chapter four

Multi-Population Mathematical Models

4-1 Introduction

Several mathematical models about the growth of diseases such as tumor growth, cancer, AIDS, etc. have been presented in researchers' research. These models are mostly based on a differential equation system and less control factor is used in them. Investigating and using the control factor is one of the main concerns of applied mathematics scientists and medical researchers. In recent years, these models have been addressed from the point of view of optimal control and have brought good results. [43] Considering that the presentation of such models is increasing in recent researches and also because these models are obtained from the results of real investigations and experiments on humans, their study is of great importance. has the models that exist in this field are presented as systems consisting of two or more ordinary differential equations. based on [44,45,46] we propose a mathematical model that involves the influence of the immune system and chemotherapy on tumor cells, but also the normal cells and the unavoidable effect of chemotherapy on these. we analyze the model in terms of the linear stability of the system.

4-2 The first model (control in multi-population models)

The models that have been presented so far in the research of researchers regarding the growth of diseases such as cancer, AIDS, etc. are divided into two important categories. One is the issues that are solely based on dynamic systems and the governing differential equations that identify and predict the behavior of the system. These models are mostly based on a differential equation system and less optimal control has been used in them. There are other problems that, in addition to being based on differential equations, the control factor is also used in them. In such models, not only the behavior of the system can be investigated without the control agent, but also with the presence of the control agent and the appropriate control, the desired results can be achieved if possible. Investigating and using the control factor is one of the main concerns of applied mathematics scientists and medical researchers [57]. Due to the fact that the presentation of such models in recent researches is increasing and also because these models are obtained from the results of real investigations and experiments on humans, their study is of great importance. Issues such as stability, controllability and observability in these systems are challenging and research issues.

Studying the treatment of human diseases from a mathematical point of view has become very important due to the incalculable cost of human lives. New methods of medical treatment such as chemotherapy, radiation therapy, immunotherapy or combined treatments with appropriate doses of patients have opened the feet of mathematicians to this issue. The mutual interaction of the medical treatment team with mathematicians can strengthen the interdisciplinary activity in this field. Due to the fact that several variables play a role in the interaction and strengthening and weakening of these diseases, therefore, based on measurable variables, the models are presented as one-population, two-population, three-population, etc.

For example, in a two-population model related to cancer treatment, where $E(t)$ represents immune cells that destroy tumor cells, $T(t)$ represents tumor cells, and t represents the independent variable of time, with two differential equations, i.e., $dE\ /dt$ and dT/dt are introduced. In such two-population models, due to the delay between chemotherapy drug injection and tumor response, or other factors, it is not able to justify the observed effect, and we have to introduce new variables to be more consistent with reality. For example, here are the new variables of normal cell population and drug concentration, which are formed by adding three-population and four-population models to the above two-population system. One of those who has had a valuable activity in the field of mathematical modeling in medical treatment is "De Pilis". De Pillis et al [57-62] investigated the three-population model and the six-population model of cancer and investigated the effect of IL-2 (interleukin-2) on tumor growth. These populations include tumor cells, specific and non-specific immune cells, and concentration of drug treatments. They investigated the interaction between treatments and came to the conclusion that for small tumors, TIL (tumor infiltrating lymphocyte) treatment, chemotherapy for discoverable tumors, and a combination of TIL and chemotherapy for large tumors. The initial mathematical models presented by De pilis are mostly based on differential equations and the control factor is not used in them. Recently, he also used the control factor in his work and investigated the interaction of different treatments available for cancer and their effectiveness in people. The following three-population differential equation system was presented by de Pilis for the growth of a tumor.

$$\begin{cases} \frac{dT}{dt} = aT(1-bT) - cNT - D \\ \frac{dN}{dt} = \sigma - fN + \frac{gT^2}{h+T^2}N - pNT \\ \frac{dL}{dt} = -mL + \frac{jD^2}{k+D^2}L - qLT + rNT \\ D = d\,\frac{(L/T)^{\lambda}}{s+(L/T)^{\lambda}}T \end{cases} \tag{4.1}$$

Where $T(t)$ represents the population of tumor cells at time $t, N(t)$ represents NK cells at time t, and $L(t)$ represents $CD8^{+}$cells cells at time t.

In this model, the control factor is not seen and this model only shows the behavior of the tumor growth system. In [63], Najarian et al. studied a three-population model with phase variables that included susceptible, treated, and AIDS patients. M Roshanfekr [64] investigated the two-population model based on healthy T cells and free viruses. Their goal is to minimize the cost and maximize the immune cells, and they used the size method to solve the optimal control problems. In the following model, we have intervened the control factor in the Pilis model. This three-population model and using optimal control in medical treatment. We present the following model for immunotherapy with the presence of two control factors. $\mathrm{u}_{N\,(t)}$ means the drug prescribed to boost normal cells and $u_{L(t)}$ means the drug prescribed to boost $CD8^{+}$cells.

$$\begin{cases} \dot{T} = aT(1-bT) - cNT - D \\ \dot{N} = \sigma - fN + \frac{gT^2}{h+T^2}N - pNT + \mathrm{u}_{N\,(t)} \\ \dot{L} = -mL + \frac{jD^2}{k+D^2}L - qLT + rNT + u_{L(t)} \\ D = d\frac{(L/T)^{\lambda}}{s+(L/T)^{\lambda}}T \end{cases} \tag{4.2}$$

The goal is to use the least number of drugs to control the growth of cancerous tumors.

$$\begin{aligned} &Min{:}J(u) = \int_{t_0}^{t_f}(T^2 + u_{T(t)}^2 + u_{N(t)}^2 + u_{L(t)}^2)dt \\ &\dot{T} = aT(1-bT) - cNT - D \\ &\dot{N} = \sigma - fN + \frac{gT^2}{h+T^2}N - pNT + \mathrm{u}_{N\,(t)} \\ &\dot{L} = -mL + \frac{jD^2}{k+D^2}L - qLT + rNT + u_{L(t)} \\ &D = d\frac{(L/T)^{\lambda}}{s+(L/T)^{\lambda}}T \end{aligned} \tag{4.3}$$

By observing the results obtained (4.3) in the recent research, it has been shown that this model does not have any destructive effect on CD+8 cells.

The first thing that should be done for the efficiency of a model is to check the stability of the model. Checking the stability of a linear system depends on the coefficient matrix of a system. The eigenvalues of this matrix play the main role in the stability of the system. Of course, what is important is not the size of the eigenvalue, but the sign of the eigenvalues is very important. We will discuss it further.

4-2-1 Dynamic analysis of the model

One of the most important issues that should be taken into consideration in formulating the treatment regimen of a cancer patient is the stability of the patient's condition in the event of improvement, which means that if we were able to bring the patient's condition to a level where the number of cancer cells is as minimal as possible with a variety of common treatments, how likely is it? that the cancer does not recur, it has been observed in many cases that after stopping the drug, the cancer grew again and reached a lethal level, so it seems that the examination of the Pilis model in terms of The stability of the level where the tumor has the minimum possible value (zero) is necessary. Tumor control in a mathematical sense means zeroing its population changes and in the dynamic analysis of the target It is to find the points where the tumor changes are zero, these points are known as the equilibrium points of the system are. After obtaining the equilibrium points, we should check the stability of the system near the equilibrium point let's pay to get the equilibrium points, we set the ODE equations of the model equal to zero

$$\frac{dT}{dt} = 0 \Rightarrow aT(1 - bT) - cNT - \overline{DT} = 0 \qquad (4.5)$$

$$\frac{dN}{dt} = 0 \Rightarrow \sigma - fN + \frac{gT^2}{h + T^2}N - pNT = 0 \qquad (4.6)$$

$$\frac{dL}{dt} = 0 \Rightarrow -mL + \frac{jD^2}{k + D^2}L - qLT + rNT = 0 \qquad (4.7)$$

$$D = \overline{D}T$$

A factor is taken from equation (4.5) in relation to T, so there have it

$$T(a(1 - b) - cN - D) = 0 \qquad (4.8)$$

Then

$$T = 0 \; and \; a(1 - b) - cN - D$$

If $T = 0$ and by substituting in (4.6)

$$\sigma - fN = 0 \Rightarrow N = \frac{\sigma}{f}$$

If $T = 0$ and by substituting in (4.7)

$$-mL = 0 \Longrightarrow l = 0$$

For the system matric Jacobi

$$\begin{bmatrix} a - 2abT - cN + A & -cT & B \\ \dfrac{2gThN}{(h+T^2)^2 - pN} & -f + \dfrac{gT^2}{h+T^2} & 0 \\ C - q + rN & rT & -m + F - qT \end{bmatrix}$$

$$A = \frac{-dT^{\lambda}}{(sT^{\lambda} + L^{\lambda})} + \frac{\lambda s dT^{\lambda}L^{\lambda}}{(sT^{\lambda} + L^{\lambda})^2}$$

$$B = (-\lambda s \left(\frac{T}{L}\right)^{\lambda - 1} \left(-\frac{T}{L^2}\right))(\frac{-dT}{(s\left(\frac{T}{T}\right)^{\lambda} + 1)^2}$$

$$C = \frac{jL\mathrm{k}\mu_1}{(k + D^2)^2}$$

$$F = \frac{jD^2}{(k + D^2) + \dfrac{Lk\mu_2}{(k + D^2)^2}}$$

$$\mu_1 = \frac{2d^2TL^{2\lambda}}{(sT^{\lambda} + L^{\lambda})^2} - \frac{2\left(\lambda s^2 d^2 T^{2\lambda} L^{2\lambda} + \lambda s d^2 T^{\lambda+1} L^{3\lambda}\right)}{(sT^{\lambda} + L^{\lambda})^4}$$

$$\mu_1 = \frac{2\lambda d^2 T^2\left(\lambda s^2 L^{2\lambda - 1} T^{2\lambda} + \lambda s T^{\lambda} L^{3\lambda - 1}\right)}{(sT^{\lambda} + L^{\lambda})^4}$$

The Jacobian matrix at the equilibrium point $T = 0$ as follows

$$\begin{bmatrix} a - cN - d & 0 & 0 \\ -pN & -f & 0 \\ -qL + rN & 0 & -m \end{bmatrix}$$

Considering that matrix is a lower triangular matrix, therefore, the eigenvalues of this matrix are equal to the values of the principal diagonals.

$$e_1 = a - cN - d \qquad e_2 = -f \quad e_3 = -m$$

Example 5.1

if you have table 1 that shows parameters of vaccine therapy model, Show whether the model stable or unstable.

Parameters	Units	Value estimated	Description
a	day^{-1}	$5.14e^{-1}$	Tumor growth rate
c	$cell^{-1}day^{-1}$	$3.23e^{-7}$	Fractional tumor cell kill by NK cells
d	day^{-1}	5.8	Saturation level of fractional tumor cell kill by $CD8^{+}T\ cells$
f	day^{-1}	$4.12e^{-12}$	Death rate of NK cells
m	day^{-1}	$2e^{-2}$	Death rate of $CD8^{+}T$cells

Solution: The Jacobian matrix at the equilibrium point $T = 0$ as follows

$$\begin{bmatrix} a - cN - d & 0 & 0 \\ -pN & -f & 0 \\ -qL + rN & 0 & -m \end{bmatrix}$$

Then

$$\begin{bmatrix} 5.14e^{-1} - 3.23e^{-7}N - 5.8 & 0 & 0 \\ -pN & -4.12e^{-12} & 0 \\ -qL + rN & 0 & -2e^{-2} \end{bmatrix}$$

$$e_1 = 5.14e^{-1} - 3.23\ e^{-7}(5 * 10^6) - 5.8 \quad e_2 = -4.12e^{-12} \quad e_3 = -2e^{-2}$$

The system is stable because $e_1, e_2\ and\ e_3$ is negative.

4-3 the second model (control in multi-population models)

Although there are different mathematical models that are used to describe tumor growth, we consider the logistic model Based on the [44],[45], let us consider N_1 the number of tumor cells, N_2 the number of normal cells, I the number of cells of the immune system, and Q the amount of chemotherapeutic drug. Our proposed model is given by:

$$\begin{cases} \frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - \frac{\alpha_{12} N_2}{K_1}\right) - c_1 I N_1 - \frac{\mu N_1 Q}{a+Q} \\ \frac{dN_2}{dt} = r_2 - \alpha_{21} N_2 N_1 - \frac{v N_2 Q}{b+Q} \\ \frac{dI}{dt} = s - mI + \frac{\rho N_1 I}{\gamma + N_1} - c_2 N_1 I - \frac{\delta I Q}{c+Q} \\ \frac{dQ}{dt} = q(t) - \lambda Q \end{cases} \tag{4.8}$$

where r_1 is the intrinsic growth rate, with the tumor cells carrying capacity given by k_1. The coefficients of competition between the populations N_1 and N_2 are given by α_{ij}, which measure the effects of population j on the population i $(i, j = 1{,}2)$. The parameter r_2 represents the total constant reproduction of normal cells (see [46]). The dynamics in the immune cell population is activated by the tumor population at a rate

ρ, γ being the half-saturation constant of the Michaelis-Menten functional response given by $\rho N_1 I = (\gamma + N_1)$ (see [44]), and also there is a natural death rate of immune cells given by m. As we know that some immune cells can directly eliminate tumor cells (see [44]), two more terms are in order$-c_1 I N_1 - c_2 I N_1$, the latter representing the inactivation of immune cells acting on tumor cells and the former is due to the death of tumor cells due to the action of the immune system. The term s describes a natural source of immune cells (see [50]). In order to model the amount of chemotherapy injected in the system, the function $q = q(t)$ models the infusion of the drug into the system and $\lambda$is the drug washout rate. As in [68], the response of each cell population to the chemotherapy is considered to be of a Michaelis-Menten form, with half-saturation parameters a, b and c, μm, is the treatment rate of the tumor cells; v is the mortality rate of normal cells due to treatment; and δrepresents the mortality rate of immune cells due to the chemotherapeutic drug (see also [69]).

the drug administration $q(t)$ takes two different forms:

1. Continuous administration

$q(t) = q > 0$, and the drug infusion is at a constant rate.

2. Administration in cycles

Following [4], $q(t)$ is a periodic function defined as

$$q(t) = \begin{cases} q_p > 0, & n < t \leq n + \tau \\ 0, & n + \tau < t \leq n + T \end{cases}$$

where T is the time between drug infusions,$n = 0, T, 2T, 3T, 4T, \dots, mT$,m Natural number represents the instants of administration, and τ is the time taken for infusion.

4-3-1 Stability analysis with chemotherapy

If we add chemotherapy, in order to study the full model (4.8). We have the same behavior in the subspaces $N_2 = 0$ and $I = 0$, as the situation without chemotherapy. let $N1 = 0$ and $q(t) = q$ (constant) results in the following system:

$$\begin{cases} \frac{dN_1}{dt} = 0 \\ \frac{dN_2}{dt} = r_2 - \frac{vN_2Q}{b+Q} \\ \frac{dI}{dt} = s - mI - \frac{\delta IQ}{c+Q} \\ \frac{dQ}{dt} = q - \lambda Q = h(Q) \end{cases} \tag{4.9}$$

The subspace is still invariant, since to the initial condition (N_1, N_2, I, Q) one

gets

$$Q(T) = Q_0 e^{-\lambda t} + \frac{q}{\lambda}(1 - e^{-\lambda t})$$

Since $N_1 = 0$ is the solution to the cancer cells population, then we write $g(t)$ and $f\ (t)$ as the solutions to the normal and immune cells populations, respectively. Let us consider the solution of the system of equations given by $Y(t) = (N_1(t), N_2(t), I(t), Q(t))$and W, the subspace where $N_1 = 0$. For an initial condition $p = (0, N_2(0), I(0), Q(0)) \in W$, the solution is given by

$$Y(t) = (0, g(t), f(t), Q_0 e^{-\lambda t} + \frac{q}{\lambda}\left(1 - e^{-\lambda t}\right)) \in W$$

Then, if the solution is in this subspace, it will remain there, resulting that W is invariant.

Consequently, we can discard $N_1 < 0$.

The equilibrium point of the chemotherapeutic equation is given by $Q(t) = \frac{q}{\lambda}$ and it is stable, since $\frac{\partial h}{\partial Q} = -\lambda$, where $h(Q) = q - \lambda Q$. We have another invariant subspace$W_3 \subseteq W$, given by $N_1 = 0$ and $Q = q - \lambda$. It is invariant because to $p = (0, N_2(0), I(0), \frac{q}{\lambda}) \in W_3$, the solution given by $Y(t) = (0, g(t), f(t), q/\lambda) \in W_3$. When $q = 0$ and $Q(0) = 0$, we conclude that W_3 has a similar behavior as the invariant plane without chemotherapy, where the solutions are getting

closer to the invariant line $I = \frac{s}{m}$ and $N_2 \to \infty$.As q increases, $q > 0$, there is an equilibrium point on the invariant line with coordinates

$$p_{inf} = (N_2, I, Q) = (\frac{(b\lambda + q)r_2}{qv}, \frac{s(q + c\lambda)}{mp + q\delta + cm\lambda}, \frac{q}{\lambda})$$

on the subspace with $N_1 = 0$. As N_1 does not change on the invariant subspace W in $3D$, we analyze the stability on this subspace. The Jacobian matrix is:

$$J_1 = \begin{bmatrix} -\frac{vQ}{b+Q} & 0 & \frac{vN_2Q}{(b+Q)^2} - \frac{vN_2}{b+Q} \\ 0 & -m - \frac{\delta Q}{c+Q} & \frac{\delta IQ}{(c+Q)^2} - \frac{\delta I}{c+Q} \\ 0 & 0 & \lambda \end{bmatrix}$$

Evaluating the Jacobian matrix on p_{inf} it follows:

$$J_1(p_{inf}) = \begin{bmatrix} -\frac{vq}{b\lambda+q} & 0 & -\frac{br_2\lambda^2}{q^2+bq\lambda} \\ 0 & -m - \frac{\delta Q}{c\lambda+q} & \frac{cs\delta\lambda^2}{(q+c\lambda)(q(m+\delta)+cm\lambda)} \\ 0 & 0 & \lambda \end{bmatrix} \qquad (4.10)$$

The eigenvalues of (4.10) are given by

$$\xi_1 = -\lambda < 0,$$

$$\xi_2 = -(\frac{qv}{b\lambda + q}) < 0,$$

and

$$\xi_3 = -(\frac{mc\lambda + mq + q\delta}{c\lambda + q}) < 0$$

implying that the equilibrium point is stable on this subspace. This point, p_{inf}, occurs due to the killing of normal cells by the chemotherapeutic drug. Now we study the cancer model in the full domain. The system (4.10) has five equilibrium points for $q > 0$; four of them have the following structure

$$p^* = (N_1^*, N_2^*, I^*, \frac{q}{\lambda})$$

And other is

$$p_{inf} = \left(0, \frac{r_2(q + b\lambda)}{qv}, \frac{qs + cs\lambda}{mq + q\delta + cm\lambda}, \frac{q}{s}\right)$$

The point p_{inf} represents the extinction of cancer cells, and it was analyzed in the subspace W, with $Q = q/\lambda > 0$. The Jacobian matrix is

$$J_2 = \begin{bmatrix} \bar{A} & -\frac{\alpha_{12} N_1 r_1}{k_1} & -c_1 N_1 & \frac{\mu N_1 Q}{(a+Q)^2} - \frac{\mu N_1}{a+Q} \\ -\alpha_{21} N_2 & -\alpha_{21} N_1 - \frac{vQ}{b+Q} & 0 & \frac{v N_2 Q}{(b+Q)^2} - \frac{v N_2}{b+Q} \\ \bar{B} & 0 & \bar{C} & \frac{\delta I Q}{(c+Q)^2} - \frac{\delta I}{c+Q} \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\bar{A} = -c_1 I - \frac{N_1 r_1}{k_1} + r_1\left(1 - \frac{N_1}{k_1} - \frac{\alpha_{12} N_1}{k_1}\right) - \frac{\mu Q}{a+Q},$$

$$\bar{B} = -c_2 I - \frac{I N_1 \rho}{(N_1+\gamma)^2} + \frac{I\rho}{N_1+\gamma},$$

and

$$\bar{C} = -m - c_2 N_{1-} \frac{\delta Q}{c+Q} + \frac{N_1 \rho}{(N_1+\gamma)}$$

Consequently, $J_2(p_{inf})$ *take the form:*

$$J_2(p_{inf}) = \begin{bmatrix} \bar{D} & 0 & 0 & 0 \\ \frac{-\alpha_{21} r_2 (q + b\lambda)}{qv} & -\frac{vq}{b + b\lambda} & 0 & \frac{b r_2 \lambda^2}{q^2 + bq\lambda} \\ \frac{-s(q + c\lambda)(c_2\gamma - \rho)}{\gamma(q\delta + m)(q + c\lambda)} & 0 & -\frac{mq + q\delta + cm\lambda}{q + c\lambda} & \bar{E} \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Where

$$\bar{D} = -\frac{c_1 s(q + c\lambda)}{q\delta + m(q + c\mu)} - \frac{q\mu}{q + a\lambda} + r_1 - \frac{r_1 \alpha_{12} r_2 (q + b\lambda)}{k_1 q v}$$

And

$$\bar{E} = -\frac{cs\delta\lambda^2}{(q + c\lambda)(q\delta + m(q + c\lambda))}$$

The eigenvalues of this matrix are

$$x_1 = -\lambda,$$

$$x_2 = -(\frac{qv}{q + b\lambda}),$$

$$x_3 = -(\frac{mq + q\delta + cm\lambda}{q + c\lambda}),$$

And

$$x_4 = r_1 - \left(\frac{c_1 s(c\lambda + q)}{mc\lambda + mq + \delta q}\right) - \frac{\mu q}{a\lambda + q} - \frac{\alpha r_2 (b + q)}{vq}$$

The first three eigenvalues are negative since the parameters are positive. The fourth one will be negative under the following condition:

$$r_1 < \frac{c_1 s(c\lambda + q)}{mc\lambda + mq + \delta q} - \frac{\mu q}{a\lambda + q} + \frac{\alpha r_2 (b + q)}{vq}$$

Conclusions

In light of the results of the research, the following can be concluded:

1- it is clear that the state variables (or, equivalently, the corresponding modes) of a linear system can generally be divided into the following four exclusive groups:

Case a: Controllable and Observable

Case b: Controllable but unobservable

Case c: Uncontrollable but observable

case d: Uncontrollable and unobservable.

2- There are some unstable control systems that can be made stable.

3- There are some unstable systems that cannot be made stable.

4-The stable systems not necessarily controllable.

5-The stable systems not necessarily observable.
6-The unstable systems not necessarily uncontrollable.
7-The unstable systems not necessarily unobservable.
8- we studied a three-population model with phase variables that included susceptible, treated, and AIDS patients. Investigated the two-population model based on healthy T cells and free viruses. Their goal is to minimize the cost and maximize the immune cells, and they used the size method to solve the optimal control problems. In the model, we have intervened the control factor in the Pilis model. This three-population model and using optimal control in medical treatment. We present the model for immunotherapy with the presence of two control factors.

Suggestions of furth works

In this thesis we studied the most important characteristics of linear control systems, which are controllability, observability and stability, and presented new ways to know the state of the system.

For further work the following suggestions are presented:

1-If the system is uncontrollable then how do we make it controllable and is that possible or not?

2-If the system is unobservable then how it can be converted to observable and is that possible or not?

3- Finding solutions to some unstable control systems to make them stable.

4- Providing mathematical models about the growth of diseases such as tumor growth, cancer, AIDS, and others, and studying them more widely and diagnosing whether they are controllable.

Signs

$A(t)$: Matrix

$B(x)$: Matrix

$u(t)$: The input

$y(t)$: The output

$x(t)$: State space

$N_U(u(t))$: The normal cone to U
I : The current
L: The electromotive force

E: Proportional to the rate of change of the current I

μ : Viscous damping coefficient.

q_i: Inflow rate

q_0: Outflow rate

h: Head level

A: The cross-section area of the tank

δ_h: Corresponding change in the head level

V : A coefficient dependent on the properties of liquid and the geometry of the valve

R̀ : A constant called the resistance of the valve at the point considered.

T: The transfer function or overall gain of positive feedback control system.
G: The open loop gain, which is function of frequency.
H: Is the gain of feedback path, which is function of frequency.
∂T: The incremental change in T due to incremental change in G.

$a(s)$: The characteristic polynomial of the system.

Q_p: Matrix for a controllable system.
Q_c: system controllability matrix.
Q_o: system observable matrix.
$D(s)$: characteristic polynomial.
$V(x)$: positive at all points in the state space.
$F(s)$:be an analytic function.

w_{cg}: stand for respectively.
w_{cp}: stand for respectively.
$E(t)$: Represents immune cells that destroy tumor cells.
$T(t)$: Represents tumor cells.
and $u_{L(t)}$: Means the drug prescribed to boost $CD8^{+}$cells.

r_1: The intrinsic growth rate.

r_2: Represents the total constant reproduction of normal cells.

p_{inf}: Represents the extinction of cancer cells.

J: The Jacobian matrix.

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استاد راهنما: دکتر هادی بصیرزاده		
درجه تحصیلی: دکتری	رشته: ریاضی کاربردی	گرایش: کنترول بهینه
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چکیده:

در این رساله ، شرایط پایداری، کنترل پذیری و مشاهده پذیری سیستم های کنترل خطی مورد بررسی قرار گرفته است. پایداری یک مفهوم مهم در سیستم های کنترل و کنترل بهینه است. روش های متفاوتی برای بررسی پایداری یک سیتم وجود دارد. ما با استفاده از دو روش "روت" و " لیاپانوف" روش جدیدی را برای بررسی پایداری یک سیستم خطی با استفاده از علامت مقادیر ویژه ارائه کرده ایم.

با استفاده از نرم افزار "MATLAB" الگوریتم هایی برای بررسی کنترل پذیری و مشاهده پذیری و پایداری یک سیستم خطی ارائه شده به گونه ای که برای کاربران استفاده از آن را آسان کرده است.

در فصل آخر ، دو مدل ریاضی چند جمعیتی برای کنترل بیماری سرطان آورده ایم و شرایط پایداری آن ها را بررسی کرده ایم.

دانشگاه شهید چمران اهواز
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