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FI-Rad_J-Supplemented and FI-⊕-Rad_J-Supplemented Modules

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Abstract

Assume that R is any ring with identity and M is a unitary left R −module. This work introduces the Rad_J-Supplemented module with respect to the fully invariant submodule it is denoted by (FI−Rad_J-Supplemented) and ⊕-Rad_J-Supplemented modules with respect to the fully invariant submodule (denoted by FI-⊕-Rad_J-Supplemented modules). As well as the main features of these modules in this work, and various properties of these modules. Also, we defined weakly FI − Rad_J − Supplemented modules as an extension of the FI-Rad_J − $s$upplemented modules and we explain the relationship between them using notes and examples.

Keywords: Supplemented Modules, Weakly Supplemented Modules, ⊕-Supplemented Modules, Rad_J-Supplemented Modules, ⊕-Rad_J-Supplemented Modules.

المقاسات المكملة من النمط FI−Rad$_J$ والمقاسات المكملة من النمط FI−⊕−Rad$_J$

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الخلاصة

افترض ان R حلقة ذات عنصر محايد وليكن M مقاساً ايسر معرف عليها. في هذا العمل نقدم تعريف المقاسات المكملة من النمط FI−Rad_J والمقاسات المكملة من النمط FI−⊕ − Rad_J مع الميزات الرئيسية لهذه المقاسات, بالاضافة الى الخصائص المختلفة لهذه المقاسات. كذلك نعرف المقاسات المكملة من النمط weakly FI − Rad_J ونبين العلاقة بينهما بأستخدام الامثلة والملاحظات .

1. Introduction

All modules are unitary left R-modules, and R will be used to signify any arbitrary associative ring with identity. A submodule P of M is known as a small submodule of M ($P \ll M$). If $P + E = M$ for all submodules E of M exists, we know that $M = E$, [1]. The Jacobson radical of M, indicated by $J(M)$, is the sum of all small, submodules of M, [2]. Where $M = Q + E$, then a submodule Q of a module M is referred to as a J −small ($Q \ll_J M$), such that$J(\frac{M}{E}) = \frac{M}{E}$ implies $E = M$,[3].

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Let M be a module and O, S be their submodules. If O is minimum with respect to $M = O + S$, then O is a supplement of S in M. Equivalent to $O + S = M$ when $O \cap S \ll O$; M is referred to as a supplemented module if it contains a supplement for each of its submodules [4]. There are many researchers who developed the Supplemented modules, see [5 -7].

If M is a module, X and Y are their submodules, then X is said to be $J-$ Supplemented of Y in M if $M = X + Y$ and $X \cap Y \ll_J X$. A module M is said to be $J-$ Supplemented if each submodule of M has a J$-S$upplement [3]. A module M is referred to as a ⊕-Supplemented if it has a direct summand Supplement T in M with the formulas $M = O + T$ and $O \cap T \ll T$ for every submodule O of M [8]. Let O and T are submodules of M; if $M = O + T$, where O is a direct summand of M, and $O \cap T \ll_J O$, O is referred to a ⊕-Jacobson-Supplement of T in M(or simply $\oplus-$J$-$Supplement). Where M contains a ⊕-J -Supplement in each of its submodules, M is referred to a ⊕-J-Supplemented module [9]. The symbol $Rad_J(M)$ is the sum of all J-small submodules M is called Rad_J-module if $Rad_J(M) = M$ [9]. If there is a submodule J of a module L such that $L = P + J$ and $P \cap J \leq Rad_J(J)$ for each submodule P of the module L, then L is said to be Rad_J-Supplemented [9]. When there is a direct summand L of a module M with $M = S + L$, $S \cap L \leq Rad_J(T)$ exists for each a submodule S of the module M, then M is said to be a ⊕-Rad_J-Supplemented module [9].

In this work we defined the FI-$Rad_J -$ Supplemented module to be an expansion of $Rad_J -$ Supplemented and FI-⊕-Rad_J-Supplemented module is presented as a generalization of ⊕-Rad_J-Supplemented module with some properties, and we see that the ⊕-Rad_J-Supplemented modules are undoubtedly FI-⊕-Rad_J-Supplemented modules. Also, it was weakly-FI-Rad_J-Supplemented have been introduced, as well as we explained the relationship between them using notes and examples.

2.FI-Rad$_J$-SUPPLEMENTED MODULES.

As a broadening of Rad_J-Supplemented modules, FI-Rad_J-Supplemented modules are introduced in this section. Remarks and properties are used to demonstrate the notion.

Recall that a submodule S of the module C is called fully invariant (or FI-submodule) if $\gamma(S) \leq S$ for each $\gamma \in End(C)$. An R-module C is called duo module if each submodule of C is fully invariant, [10].

Definition 2.1: For every FI-submodule P of W there exists a submodule U of a module W with $W = P + U$ and $P \cap U \leq Rad_J(U)$, then W is said to be fully invariant -Rad_J-Supplemented (dented by FI- Rad_J-Supplemented).

Examples and Remarks 2.2:

1. Each semi-simple R-module is an FI-Rad_J-Supplemented. To demonstrate that, given M as a semi-simple module and T is an FI-submodule in M, there occurs a submodule V of M with $T + V = M$, $T \cap V = T \leq Rad_J$(V). So, the $\mathbb{Z}_6$ as a $\mathbb{Z}-$ module is an FI$-Rad_J -$Supplemented.
2. Each module that is $Rad_J -$Supplemented is also an FI$-Rad_J - s$upplemented.
3. The Q as a $\mathbb{Z} -$module is an FI-$Rad_J -$Supplemented module where Q is the set of rational number, since $Rad_J(Q) = Q$ and the only two $\{0\}$ and Q are fully invariant submodules of Q.
4. Each Rad_J-module is a FI$-Rad_J -$Supplemented module. But the converse need not be true. For example, the $\mathbb{Z}_4$ as a $\mathbb{Z}$ -module is an FI$-Rad_J -$Supplemented but not a Rad_J-module.

Next, we will study some properties of an FI-Rad_J −Supplemented modules, including the following:

Proposition 2.3: If Q is any FI-submodule of an FI−Rad_J −Supplemented module W, then $\frac{W}{Q}$ is an FI−Rad_J −Supplemented.

Proof: Assume that $\frac{S}{Q}$ is an FI-submodule of $\frac{W}{Q}$, so S is also fully invariant of W, [10]. Because W is a FI − Rad_J − Supplemented, then the submodule H of W exists such that $W = S + H$ with $S \cap H \leq Rad_J(H)$. So, $\frac{W}{Q} = \frac{S+H}{Q} = \frac{S}{Q} + \frac{H+Q}{Q}$, and $\frac{S}{Q} \cap \frac{H+Q}{Q} = \frac{S\cap(H+Q)}{Q} = \frac{(S\cap H)+Q}{Q}$, by Modular Law and since $S \cap H \leq Rad_J(H)$. Hence, $\frac{(S\cap H)+Q}{Q} \leq \frac{Rad_J(H)+Q}{Q}$ and $\frac{Rad_J(H)+Q}{Q} \leq Rad_J(\frac{H+Q}{Q})$ by [9]. Therefore, $\frac{W}{Q}$ is an FI−Rad_J −Supplemented.

Proposition 2.4: If O and W_1 are fully invariant submodules of the module W. Where W_1 is an FI−Rad_J − Supplemented module with $W_1 + O$ has an FI−Rad_J − Supplement W, then O is also has an FI−Rad_J −Supplement in W.

Proof: There exists an FI-submodule J of W such that $J + (W_1 + O) = W$and $J \cap (W_1 + O) \leq Rad_J(J)$, since $W_1 + O$ has an FI−Rad_J −Supplement in W, now W_1 is an FI−Rad_J −Supplemented module means that $(J + O) \cap W_1$ has a submodule V, such that $(J + O) \cap W_1 + V = W_1$ and $(J + O) \cap V \leq Rad_J(V)$. Thus $W = W_1 + O + J = (J + O) \cap W_1 + V + O + J = V + O + J$ and $(J + O) \cap V \leq Rad_J(V)$, that is V is an FI−Rad_J −Supplement of $J + O$ in W. Also, $J + V$ is an FI−Rad_J −Supplement of O in W. It is obvious that $W = (J + V) + O$, so it suffices that $(J + V) \cap O \leq Rad_J(J + V)$. Since $V + O \subseteq W_1 + O$, then $J \cap (V + O) \subseteq J \cap (W_1 + O) \leq Rad_J(J)$ and $J \cap (V + O) \leq Rad_J(J)$. Thus $(J + V) \cap O \subseteq J \cap (V + O) + V \cap (J + O) \leq Rad_J(J) + Rad_J(V) \leq Rad_J(J + V)$.

Proposition 2.5: Let $C = M_1 \oplus M_2$. Then M_1 and M_2 are FI−Rad_J − Supplemented, modules if and only if C is an FI−Rad_J −Supplemented module.

Proof:

⟹) Assume that M_1 and M_2 are FI−Rad_J − Supplemented, and O is an FI-submodule of M. So, $C = M_1 + M_2 + O$ trivially $M_2 + O$ has FI−Rad_J −Supplement in C. So, C is an FI−Rad_J −Supplemented module because according to Proposition 2.4, O has an FI−Rad_J −Supplement in C.

(⟸) Since C is a $FI - Rad_J$ −Supplemented, module and according to Proposition 2.3, $\frac{C}{M_1}$ is also an FI−Rad_J −Supplemented module, $M_2 \cong \frac{C}{M_1}$. So, M_2 is the FI−Rad_J −Supplemented module. Similarly, M_1 is an FI−Rad_J −Supplemented module.

Corollary 2.6: If J is an FI−Rad_J −Supplement of T in M_1 and W is the duo module such that $W = M_1 \oplus M_2$, then $J \oplus M_2$ is the FI−Rad_J −Supplement of T in W. Where J and T are fully invariant submodules of M_1.

Proof: Since J is an FI-Rad_J-Supplement of T in M_1, so $M_1 = J + T$ and $J \cap T \leq Rad_J(J)$. So, $W = M_1 \oplus M_2$, then $W = (J + T) \oplus M_2$, hence $W = T + (J \oplus M_2)$ but $(J \oplus M_2) \cap T = (J \oplus M_2) \cap M_1 \cap T = J \cap T \leq Rad_J(J)$ and since $J \leq J \oplus M_2$, so $Rad_J(J) \leq$

$Rad_J(J \oplus M_2)$ [9]. Hence, $J \cap T \leq Rad_J(J \oplus M_2)$. Therefore, $J \oplus M_2$ is an FI$-Rad_J$ $-$Supplement, of T in W.

Proposition 2.7: Suppose that W is an R-module, S and U are fully invariant submodules of W. If S is an FI$-Rad_J$ $-$Supplement of U in W, hence $\frac{S+J}{J}$ is an FI$-Rad_J$ $-$Supplement, of $\frac{U}{J}$ in $\frac{W}{J}$, where $J \subseteq U$.

Proof: Since S is an FI$-Rad_J$ $-$Supplement of U in W. Hence, $W = U + S$, $H \cap S \leq Rad_J(S)$, then $\frac{W}{J} = \frac{U+S}{J} = \frac{U}{J} + \frac{S+J}{J}$. Claim that $\frac{U}{J} \cap \frac{S+J}{J} \leq Rad_J(\frac{S+J}{J})$; since $\frac{U}{J} \cap \frac{S+J}{J} = \frac{U \cap (S+J)}{J} = \frac{(U \cap S)+J}{J}$ by modular law, and since $U \cap S \leq Rad_J(S)$, hence $\frac{(U \cap S)+J}{J} \leq \frac{Rad_J(S)+J}{J}$. But $\frac{Rad_J(S)+J}{J} \leq Rad_J(\frac{S+J}{J})$[9]. Hence $\frac{S+J}{J}$ is an FI$-Rad_J$ $-$Supplement of $\frac{U}{J}$ in $\frac{W}{J}$.

Proposition 2.8: Let C be an R-module and E be an FI$-Rad_J$ $-$Supplement of L in C, with F and O are fully invariant submodules of E. Then O is an FI$-Rad_J$ $-$Supplement of F in E if and only if O is an FI$-Rad_J$-Supplement, of $L + F$ in C.

Proof :

$\Longrightarrow$) Suppose that O is an FI$-Rad_J$ $-$Supplement of F in E, then $E = O + F$, $U \cap F \leq Rad_J(O)$, let $(L + F) + W = M$ for $W \subseteq O$ with $J(\frac{O}{W}) = \frac{O}{W}$. Now, $F + W \subseteq E$. Since $\frac{E}{F+W} = \frac{O+(F+W)}{F+W} \cong \frac{O}{O \cap (F+W)} = \frac{O}{W+(F \cap O)}$ by according to Second Isomorphism and Modular Law, with $J(\frac{O}{W}) = \frac{O}{W}$ [3]. Therefore, $J(\frac{O}{W+(F \cap O)}) = \frac{O}{W+(F \cap O)}$, hence $J(\frac{E}{F+W}) = \frac{E}{F+W}$. Since E is FI$-Rad_J$ $-$Supplement of L in C, so $C = L + E$, hence $E = F + W$, since $E = O + F$, $W \subseteq O$ and $J(\frac{O}{W}) = \frac{O}{W}$; thus $O = W$, [3]. Hence O is $J-$Supplemented of $L + F$ in M, then $C = (L + F) + O$ and $(L + F) \cap O \ll_J W$, then $(L + F) \cap O \leq Rad_J(O)$. Therefore, O is an FI$-Rad_J$ $-$Supplement of $L + F$ in C .

$\Longleftarrow$) Let O be an FI-Rad_J- Supplement of $L + F$ in C. Then $O + (L + F) = C$ and $O \cap (L + F) \leq Rad_J(O)$. Let $O + F = E$ to prove $O \cap F \leq Rad_J(O)$, since $O \cap F \subseteq O \cap (L + F) \leq Rad_J(O)$, then $O \cap F \leq Rad_J(O)$. Hence, V is an FI$-Rad_J$ $-$Supplemented of F in E.

Definition 2.9: If P is an FI$-Rad_J - S$upplement of Y in C and Y is an FI$-Rad_J - s$upplement of P in C such that P and Y are fully invariant submodules of the module C, then P and Y are called mutual, FI$-Rad_J -$ Supplements.

Corollary 2.10: Let C be an R–module, S and T are mutual FI$-Rad_J$ $-$Supplemen in C, and let J be an FI$-Rad_J$ $-$Supplemented of L in S and E be an FI$-Rad_J$ $-$Supplement of F in T. So, $J + E$ is an FI$-Rad_J$ $-$Supplement of $F + L$ in C.

Proof: From the hypothesis T is an FI$-Rad_J$ $-$Supplemented of S in C, and E is an FI$-Rad_J$ $-$Supplement of F in T. Hence, by Proposition 2.8, E is an FI$-Rad_J$ $-$Supplemented of $S + F$. Since $S = J + L$ then$E$ is FI$-Rad_J -$ Supplemented of $J + L + F$ in C, so $(L + J + F) \cap E \leq Rad_J(E)$. Since $T = E + F$ and S is an FI$-Rad_J -$ Supplement of T in C, and J is an FI$-Rad_J -$ Supplement of L in S, so by Proposition 2.8, J is an FI$-Rad_J$ $-$Supplement of $E + F + L$ in C. Hence, $(E + F + L) \cap J \leq Rad_J(J)$, because $C = T + S$, hence we have $C = E + F +$ J $+L = J + E + F + L$, so $(J + E) \cap (F + L) \leq (L + J + F) \cap E + J \cap$

$(E + F + L) \leq Rad_J(E) + Rad_J(J) \leq Rad_J(E + J)$, [9]. Since J and E are fully invariant in C, then $J + E$ is fully invariant in C, [11]. Thus, $J + E$ is an FI$-Rad_J$ $-$Supplement of $F + L$ in C.

3. FI-⊕-$\mathbf{Rad}_J$ −SUPPLEMENTED and WEAKLY-FI-$\mathbf{Rad}_J$-SUPPLEMENTED MODULES.

In this part, we introduced ⊕-Rad_J-Supplemented with respect to fully invariant submodules as an extension of the ⊕-Rad_J $-$Supplemented, as well as introducing the weakly$-$FI $- Rad_J$ $-$Supplemented modules as an extension of FI-Rad_J $-$Supplemented and examining some of their characteristics.

Definition 3.1: If there is a direct summand U of a module C with $C = V + U$ and $V \cap U \leq Rad_J(U)$ for each FI-submodule V of the module C, then C is called FI$-\oplus-Rad_J$ $-$Supplemented.

Examples 3.2:
(1) Q as $\mathbb{Z}$ $-$module is an FI$-\oplus - Rad_J$ $-$Supplemented, since $Rad_J(Q) = Q$ and the only direct summand are Q and $\{0\}$.
(2) Each module that is $\oplus - Rad_J$ $-$Supplemented is also FI$-\oplus - Rad_J$ $-$Supplemented.

An R-module C is referred to as an FI-⊕-J -Supplemented if for each FI-submodule S of Cthere exists a direct summand U of C such that $C = S + U$ and $S \cap U \ll_J U$.

Proposition 3.3: Suppose that W is a module with $Rad_J(W) \ll_J W$. Then W is FI-⊕-J-Supplemented if and only if W is a FI$-\oplus - Rad_J$ $-$Supplemented module.
Proof:
$\Rightarrow$) Clear.
$\Leftarrow$)Assume that X is an FI-submodule of W. W is FI$-\oplus - Rad_J$ $-$Supplemented, hence a direct summand Y of W exists such that $W = X + Y$, and $X \cap Y \leq Rad_J(Y)$ additionally, since $Y \leq W$, $Rad_J(Y) \leq Rad_J(W)$, [9] and since $Rad_c(W) \ll_J W$, hence $Rad_J(Y) \ll_J W$ [3]. Also, since $X \cap Y \leq Rad_J(Y)$, then $X \cap Y \ll_J W$. Y is a direct summand of W . So $X \cap Y \ll_J Y$[3]. Therefore, W is FI-⊕-J $-$Supplemented module.

Proposition 3.4: If L is an FI-submodule of M and M is an FI$-\oplus - Rad_J$ $-$Supplemented module, then $\frac{M}{L}$ is FI$-\oplus - Rad_J$ $-$Supplemented.
Proof: Assume that $\frac{T}{L}$ be an FI-submodule of $\frac{M}{L}$. So, T is an FI-submodule of M. Because M is FI$-\oplus - Rad_J$ $-$Supplemented, there is a direct summand P of M with $M = T + P$, $T \cap P \leq Rad_J(P)$, and $M = P \oplus P'$, $P' \subseteq M$, so $\frac{M}{L} = \frac{T+P}{L} = \frac{T}{L} + \frac{P+L}{L}$, $\frac{T}{L} \cap \frac{P+L}{L} \leq Rad_J(\frac{P+L}{L})$, $\frac{T}{L} \cap \frac{P+L}{L} = \frac{T \cap (P+L)}{L} = \frac{(T \cap P)+L}{L}$, by Modular Law. Because $T \cap P \leq Rad_J(P)$, then $\frac{(T \cap P)+L}{L} \leq \frac{Rad_J(P)+L}{L}$ and $\frac{Rad_J(P)+L}{L} \leq Rad_J(\frac{P+L}{L})$ [9]. Since $M = P \oplus P'$ and L is an FI-submodule of M, then $\frac{M}{L} = \frac{P+L}{L} \oplus \frac{P'+L}{L}$. Hence $\frac{M}{L}$ is FI$-\oplus - Rad_J$ $-$Supplemented.

Proposition 3.5: Assume W be a module such that T and $W_1$are fully invariant submodules of W with W_1 be a FI$-\oplus - Rad_J - S$upplemented module, if $W_1 + T$ has FI$-\oplus - Rad_J$ $-$Supplement in W, so T has an FI$-\oplus - Rad_J$ $-$Supplement in W.

Proof: From the hypothesis, then there is a direct summand O of W, with $O + (W_1 + T) = W$ and $O \cap (W_1 + T) \leq Rad_J(O)$. Also, there exists a direct summand J of W_1, with $(O + T) \cap W_1 + J = W_1$ and $(O + T) \cap J \leq Rad_J(J)$. It follows that $W = W_1 + T + O = J + T + O$, and $(O + T) \cap J \leq Rad_J(J)$, which is J is $\oplus - Rad_J$ −Supplement of $O + T$ in W. Since $W = (O + J) + T$ and according to Proposition 2.4, we have $(O + J) \cap T \leq Rad_J(O + J)$. Therefore, $O + J$ is an FI−$\oplus - Rad_J$ −Supplement of T in W.

Proposition 3.6: Assume that $W = W_1 \oplus W_2$, then W_1 and W_2 are FI−$\oplus - Rad_J$ −Supplemented modules if and only if W is an FI−$\oplus - Rad_J$ −Supplemented module. Where W_1 is an FI-submodule of W.
Proof: Clear according to Proposition 2.5.

Corollary 3.7: If D is an FI−$\oplus - Rad_J$ −Supplement of J in M_1, then $D \oplus M_2$ is an FI−$\oplus - Rad_J$ −Supplement of J in M. Where $M = M_1 \oplus M_2$ is a duo module, D and J are fully invariant submodules of M_1.
Proof: Clear by Corollary 2.6.

Proposition 3.8:
Let C = $C_1 \oplus C_2$ be a duo module, and U, O be FI-submodules of C_1, if U is an FI $- \oplus$-Rad_J-Supplement of O in C_1, then U $\oplus C_2$ is an FI $- \oplus$-Rad_J- Supplement of O in C.
Proof:
Let U be an FI $- \oplus$-Rad_J-Supplement of O in C_1, then C_1 = U + O and U ∩ O $\leq Rad_J(U)$. Since C = $C_1 \oplus C_2$, then C = (U + O) $\oplus C_2$, hence C = O+ (U $\oplus C_2$), but (U $\oplus C_2$) ∩ O = (U $\oplus C_2$) ∩ C_1 ∩ O = U ∩ O $\leq Rad_J(U)$ and as U $\leq U \oplus C_2$, then $Rad_J(U) \leq Rad_J(U \oplus C_2)$, by [9]. Hence, U ∩ O $\leq Rad_J(U \oplus C_2)$. Therefore, U $\oplus C_2$ is an FI $- \oplus$-Rad_J-Supplement of O in C.

Theorem 3.9: Suppose that M is an R-module where $M = M_1 \oplus M_2$ is a direct summand of submodules M_1 and M_2. Then M_2 is an FI−$\oplus - Rad_J$ − Supplemented module if and only if there exists a direct summand J of M with $J \subseteq M_2$, $M = J + T$, and $J \cap T \leq Rad_J(J)$ for each FI-submodule $\frac{T}{M_1}$ of $\frac{M}{M_1}$.
Proof:
⟹) Suppose that $\frac{T}{M_1}$ be an FI-submodule of $\frac{M}{M_1}$. Then T ∩ M_2 is an FI-submodule of M_2. Since M_2 is a FI−$\oplus - Rad_J$ −Supplemented module, so T ∩ M_2 has a direct summand J of M_2 with $M_2 = (T \cap M_2) + J$ and $T \cap M_2 \cap J = T \cap J \leq Rad_J(J)$. As J is the direct summand of M with $M = M_1 + M_2 = M_1 + (T \cap M_2) + J \subseteq M_1 + T + J$. However, $M_1 \subseteq T$. So, $M = J + T$.
⟸) Let T be an FI-submodule of M_2, therefore $\frac{T \oplus M_1}{M_1}$ is an FI-submodule of $\frac{M}{M_1}$. According to our presumption, a direct summand J of M there exists with $J \subseteq M_2$, $M = (T + M_1) + J$, $(T + M_1) \cap J \leq Rad_J(J)$. Since $M_2 = M_2 \cap M = M_2 \cap [(T + M_1) + J] = J + [(T + M_1) \cap M_2] = J + T + (M_1 \cap M_2) = J + T$, by Modular Law and since $J \cap T \subseteq (T + M_1) \cap J \leq Rad_J(J)$, then $J \cap T \leq Rad_J(J)$. Therefore, M_2 is an FI−$\oplus$-Rad_J-Supplemented module.

Proposition 3.10: Assume that N is an R-module. If E has an FI−$\oplus$-Rad_J − Supplement in N. Then $\frac{E}{G}$ has an FI−$\oplus$-Rad_J −Supplement in $\frac{N}{G}$, where G is a fully invariant of N such that, G $\subseteq E$.

Proof: Form the hypotheses E has an FI$-\oplus$-Rad_J $-$Supplement in N, so there is a direct summand S of N, with $S+E=N$, $S\cap E\ \leq Rad_J(S)$, and $N=S\oplus S'$, where $S'\subseteq N$. Then $\frac{N}{G}=\frac{E}{G}+\frac{S+G}{G}$, $\frac{E}{G}\cap\frac{S+G}{G}=\frac{E\cap(S+G)}{G}=\frac{(S\cap E)+G}{G}$ by Modular Law, and since $S\ \cap\ E\ \leq Rad_J(S)$, hence $\frac{(S\cap E)+G}{G}\leq\frac{Rad_J(S)+G}{G}$, and $\frac{Rad_J(S)+G}{G}\leq Rad_J(\frac{S+G}{G})$ by [9]. Therefore, $\frac{E}{G}$ has an FI-$\oplus$-Rad_J-Supplement in $\frac{N}{G}$.

Proposition 3.11: If V and L are FI-submodules of an R-module M, with L is a FI$-\oplus-Rad_J$ $-$Supplement of V in M and $J\ll_J M$, then L is an FI$-\oplus-Rad_J$ $-$Supplemented of $V+J$.

Proof: From the hypotheses $M=L+V$, L is a direct summand of M, and $L\cap V\leq Rad_J(L)$. So, $L+(V\ +\ J)=M$. Let $L\cap(V\ +\ J)\ +\ T=L$, where $T\leq L$ and $J(\frac{L}{T})=\ \frac{L}{T}$. Hence, $M=L+(V+J)=L\cap(V+J)+T+(V+J)=(V+T)+J$, so $\frac{M}{V+T}=\frac{L+(V+T)}{(V+T)}\cong\frac{L}{L\cap(V+T)}=\frac{L}{T+(V\cap L)}$ by Second Isomorphism Theorem and by Modular Law. Because $J\ (\frac{L}{T})=\frac{L}{T}$, we get $J\ (\frac{L}{T+(V\cap L)})=\frac{L}{T+(V\cap L)}$ [3]. Thus, $J(\frac{M}{V+T})=\frac{M}{V+T}$, and because $J\ll_J$ M so $M=\ V+\ T$, but $M=V+H$, and $T\ \subseteq\ L$ with $J(\frac{L}{T})=\frac{L}{T}$, so $L=T$, hence $L\cap(V+J)\ll_J$ L , $L\cap(V\ +\ J)\leq Rad_J(L)$. Hence, L is FI$-\oplus-Rad_J$ $-$Supplement of $V+J$ in M.

Definition 3.12: A module M is said to be a weakly$-$FI $-\ Rad_J$ $-$Supplemented if there is a fully invariant G of M with $M=T+G$ and $T\cap G\leq Rad_J(M)$ for each submodule T of M.

Remark 3.13: Every FI-Rad_J $-$Supplemented module is weakly$-$FI $-\ Rad_J$ $-$Supplemented.

Proposition 3.14: If E is a non-zero weakly$-$FI $-\ Rad_J$ $-$Supplemented module and F is a FI-submodule of E, so $\frac{E}{F}$ is a weakly$-$FI $-\ Rad_J$ $-$Supplemented.

Proof: Assume $\frac{L}{F}$ be an FI-submodule of $\frac{E}{F}$, so L is a fully invariant of E[10]. Given that E is a weakly$-$FI $-\ Rad_J$ $-$Supplemented, so there is an FI-submodule S of E with $E=L+S$, $L\cap S\leq Rad_J(E)$. Then $\frac{E}{F}=\frac{L+S}{F}=\frac{L}{F}+\frac{S+F}{F}$, $\frac{L}{F}\cap\frac{S+F}{F}\ \leq Rad_J(\frac{S+F}{F})$, since $\frac{L}{F}\cap\frac{S+F}{F}=\frac{L\cap(S+F)}{F}=\frac{(L\cap S)+F}{F}$, by Modular Law and since $L\cap S\leq Rad_J(S)$, hence $\frac{(L\cap S)+F}{F}\leq\frac{Rad_J(S)+F}{F}$ and $\frac{Rad_J(S)+F}{F}\leq Rad_J(\frac{S+F}{F})$ .Then $\frac{E}{F}$ is an FI-Rad_J–Supplemented. Therefore, $\frac{E}{F}$ is a weakly$-$FI $-\ Rad_J$ $-$Supplemented.

Corollary 3.15: If U is a weakly$-$FI $-\ Rad_J$ $-$Supplemented duo module, so each factor of modules of M is also a weakly$-$FI $-\ Rad_J$ $-$Supplemented module.

Proposition 3.16: Let M and M_1 be weakly$-$FI $-\ Rad_J$ $-$Supplemented modules with M_1, $C\subseteq M$. Then, if $M_1+\ C$ has a weakly$-$FI $-\ Rad_J$ $-$Supplement in M, so C has a weakly$-$FI $-\ Rad_J$ $-$ Supplement in M

Proof: From the hypotheses M_1+C has a weakly$-$FI $-\ Rad_J$ $-$Supplement in M, so there is an FI-submodule F of M with $F+(M_1+C)=M$, and $F\ \cap\ (\ M_1+\ C\ )\leq Rad_J(M)$. Now, since M_1 is a weakly$-$FI $-\ Rad_J$ $-$Supplemented module and $(F+C)\ \cap M_1\leq M_1$. So, there is an FI-submodule T of M_1 such that $(F+C)\cap M_1+T=M_1$ and $(F+C)\ \cap\ T\leq Rad_J(M_1)$. Thus, we have $M=M_1+C+F=(F+C)\cap M_1+T+C+F=T+C+F$, and $(F+C)\cap T\leq Rad_J(M_1)$, so $(F+C)\cap T\leq Rad_J(M)$, that is T is a weakly -FI-Rad_J-Supplement of $F+C$ in M. It is obvious that $M=(F+T)+C$, hence it suffices to demonstrate that $(F+T)\cap C\leq Rad_J(M)$. Since $T+C\subseteq M_1+C$, then $F\cap(T+C)\subseteq F\cap(M_1+C)\leq Rad_J(M)$. Then $F\cap(T+C)\leq Rad_J(M)$. Thus, $(F+T)\cap C\subseteq F\cap(T+C)+T\cap(F+C)\leq Rad_J(M)$. Therefore, C has a weakly$-$FI$-Rad_J-$Supplement in M.

Proposition 3.17: If $U=U_1\oplus U_2$ which is a duo module, then U_1 and U_2 are weakly$-$FI$-Rad_J-$ Supplemented modules if and only if U is a weakly$-$FI$-Rad_J-$Supplemented.

Proof:
$\Rightarrow$)Assume that U_1 and U_2 are weakly$-$FI$-Rad_J-$Supplemented modules. To demonstrate that U is weakly$-$FI$-Rad_J-$Supplemented. Let L be a submodule of U, since $U=U_1+U_2+L$, so trivial U_2+L has a weakly$-$FI$-Rad_J-$Supplement in U. According to Proposition 3.16, L has a weakly$-$FI$-Rad_J-$Supplemented in U, hence U is a a weakly$-$FI$-Rad_J-$Supplemented module.
$\Leftarrow$)Suppuse that U is a weakly$-$FI$-Rad_J-$Supplemented, to demonstrate that U_1 and U_2 are weakly$-$FI$-Rad_J-$Supplemented modules. Since $U_2\cong\frac{U}{U_1}$ and U is weakly$-$FI$-Rad_J-$Supplemented, so $\frac{U}{U_1}$ is a weakly$-$FI$-Rad_J-$Supplemented module according to Proposition 3.14. As a result, U_2 is a weakly$-$FI$-Rad_J-$Supplemented. Then U_1 is a weakly$-$FI$-Rad_J-$Supplemented module.

The following standard lemma is required to demonstrate that a finite sum of a weakly$-$FI$-Rad_J-S$upplemented is weakly$-$FI$-Rad_J-$Supplemented.

Lemma 3.18: Assume that O and P are submodules of a module J, and $O+P$ having a weakly$-$FI$-Rad_J-$Supplemented L in J and $O\cap(L+P)$ having a weakly$-$FI$-Rad_J-$Supplement V in O. Then $L+V$ is a weakly$-$FI$-Rad_J-$Supplement of P in J.
Proof: From the hypotheses L is a weakly$-$FI$-Rad_J-$Supplement of $O+P$ in J and V is a weakly$-$FI$-Rad_J-$Supplement of $O\cap(L+P)$ in O. Then $J=(O+P)+L$, where L an FI-submoduleof J and $(O+P)\cap L\leq Rad_J(J)$, and $O=[O\cap(L+P)]+V$, where V is an FI-submodule of O and $(L+P)\cap V\leq Rad_J(\mathrm{O})$. Since $J=(O+P)+L=[O\cap(L+P)]+V+P+L=O\cap(L+P)+V+P+L=P+L+V$, then $J=P+(L+V)$ because $V+P\leq O+P$ then $(V+P)\cap L\leq(O+P)\cap L\leq Rad_J(\mathrm{J})$, hence $(V+P)\cap L\leq Rad_J(\mathrm{J})$, then $P\cap(L+V)\subseteq[(V+P)\cap L]+[(O+P)\cap V]\leq Rad_J(\mathrm{J})$. Therefore, $L+V$ is a weakly$-$FI$-Rad_J-$Supplement of P in J.

Proposition 3.19: The finite sum of a weakly$-$FI$-Rad_J-$Supplemented modules is a weakly$-$FI$-Rad_J-$Supplemented module.
Proof: Clear.

4. Conclusion

Through our research, we conclude that the Supplemented modules and the small submodules have many and diverse fields in mathematics. We advise researchers to develop this topic and formulate other topics

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