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A NEW ANALYTICAL AND NUMERICAL SOLUTIONS FOR BIOLOGICAL POPULATION MODELS BY KAMAL ADOMIAN DECOMPOSITION METHOD

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Abstract. In this study, we present analytical and numerical solutions to the nonlinear fractional biological population equation (NFBPE) with the fractional derivative Atangana Baleanu (FDAB) using the Kamal Adomian decomposition method (KADM) and the Maple soft and MATLAB. In addition, this study discussed the existence, uniqueness, and convergence of the solution of this equation. In the end, the method effectively solved the equation and we obtained new and satisfactory results.

Keywords: Kamal transform, Adomian decomposition method, biological population models, Atangana-Baleanu operator.

AMS Subject Classification: 35Axx, 35Q62.

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1 Introduction

The diffusion of biological populations is one of the most common nonlinear scientific processes, and fractional-order differential equations are becoming more and more common in many engineering applications and research fields. In many environmental, economic, and medical systems, population dynamics research and population development prediction depend heavily on the application of biology and population equations. These applications are based on a system of mathematical equations that simulate interactions between people in a population as well as biological population development (Khater, 2022; Srivastava et al., 2014; Baleanu et al., 2024). Ant biological population equations are primarily used in a range of applications. Understanding how environmental changes affect species diversity and population dynamics, enhancing agricultural resource management, and comprehending disease distribution and the variables influencing disease transmission within populations are some of the topics covered [4]. One of the most important biological population equations that will be studied in this paper is given by the following formula: (Mahdi et al., 2023; Attia et al., 2020; Jassim et al., 2024)

$$\frac{\partial}{\partial\tau}\phi(\eta,\theta,\tau)=\frac{\partial^2}{\partial\eta^2}\phi^2(\eta,\theta,\tau)+\frac{\partial^2}{\partial\theta^2}\phi^2(\eta,\theta,\tau)+\phi(\eta,\theta,\tau)-p\phi^2(\eta,\theta,\tau), \tag{1}$$

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with given initial condition

$$\phi(\eta,\theta,0)=f(\eta,\theta) \tag{2}$$

where ϕ denotes the population density and p is real numbers.

Many researchers have presented studies on the biological model of the population, and the reader can view some of these studies. In 2015, Bououden, Chadli, and Karimi presented an adaptive fuzzy control (AFC) approach for managing uncertain and highly nonlinear biological processes, specifically in wastewater pre-treatment (Bououden et al., 2015). In 2017, Akimenko developed an explicit recurrent algorithm and performed a numerical study of travelling wave solutions for two age-structured population dynamics models with nonlinear death rates and polycyclic reproduction (Akimenko, 2017). In 2018, Chun Wu and Weiguo Rui introduced a method for finding exact solutions to nonlinear time-fractional partial differential equations (PDEs), avoiding the invalid fractional chain rule. In 2020, Srivastava et al. investigated a fractional-order biological population model with carrying capacity, and authors employed blended homotopy techniques in combination with the Sumudu transform to solve the nonlinear fractional-order population model, using the Caputo-type fractional derivative (Srivastava et al., 2020). In 2022, Mostafa Khater presented precise solutions for nonlinear fractional population biology (FBP) models using the generalized Khater (GK) technique and Atangana's conformable fractional (ACF) derivative operator. The model accounts for natural death and birth rates to derive demographic insights (Mahdi et al., 2024). In 2024, Mamunur Rashid et al. examine soliton solutions with time-dependent variable coefficients in the Kolmogorov-Petrovsky-Piskunov (KPP) model, which was initially used to model the spread of advantageous genes in populations but also applies to various physics, biological, and chemical models (Roshid et al., 2024).

In this work, we use the Kamal Adomian decomposition method (KADM) to solve fractional biological population equations that include the fractional operator of the fractional operator Atangana-Baleanu-Caputo. The paper is arranged in the following way: The basic definitions for calculus and fractional integration are presented in section 2, the algorithm of the method used in section 3, the convergence of the method is discussed in section 4, and many examples are given that explain the effectiveness of the method proposed in section 4, and finally, the conclusion is provided in section 5.

2 Basic concepts

In this section, we present some definitions and properties on which this work is based in subsequent sections.

Definition 1. *(Oprzedkiewicz & Mitkowski, 2019) Suppose that ϑ is a real number such that $0<\vartheta\le 1$, then then following normalization function $\mathcal{B}(\vartheta)$ is a function satisfies the following condition $\mathcal{B}(0)=\mathcal{B}(1)=1$.*

Definition 2. *(Askey & Roy, 2010) Let ω is a complex number then the gamma function $\Gamma(z)$ is given by*

$$\Gamma(\omega)=\int_0^\infty e^{-\xi}\xi^{\omega-1}d\xi,\ \mathrm{Re}(\omega)>0.$$

One of the most important properties of the gamma function $\Gamma(\omega+1)=\omega\Gamma(\omega)$.

Definition 3. *(Rahman et al., 2022) Let's assume that ω is a complex number, then the following is a one-parameter formulation of the Mittag-Leffler function*

$$E_\omega(\xi)=\sum_{k=0}^\infty \frac{\xi^k}{\Gamma(\omega k+1)},\ \omega>0$$

Definition 4. *(Edmunds & Lang, 2023) Let us assume that $\phi(\xi)$ is a differentiable over interval $\Omega=(\mathfrak{y}_1,\mathfrak{y}_2)$, $\mathfrak{y}_1\le\mathfrak{y}_2$, then the Sobolev space is*

$$H^1(\Omega)=\{\phi\in L^2(\Omega)\big|\ \frac{\partial^i}{\partial\xi^i}\phi\in L^2(\Omega),\ i=1,2,3,\dots,n\},$$

where $L^2(\Omega)$ *Lebesgue space.*

Definition 5. *(Abed & Hussein, 2024) Let* $\phi \in H^1(\Omega)$, *then the derivative of Atangana Baleanu* $\mathcal{Q}_\tau^\vartheta$ *with fractional order* $0 < \vartheta < 1$ *is*

$$\mathrm{H}^1(\Omega) = \left\{ \phi \in L^2(\Omega) | \frac{\partial_1}{\partial \xi_i} \phi \in L^2(\Omega), \;\; i = 1, 2, 3, ..., n \right\} \tag{3}$$

where ϕ' *is the derivative of* ϕ.

Definition 6. *(Jafari, 2021) The Kamal transform (KT) is defined over the set of function*

$$\mathcal{A} \;=\; \left\{ \phi(\tau) \;\middle|\; \exists \mathcal{M}, z_1 \;, z_2 > \;\; 0, |\phi(\tau)| < \mathcal{M} e^{\frac{|\tau|}{z_j}} \;,\; if \;\, \tau \in (-1)^j \times [0, \infty) \right\},$$

by

$$^K T[\phi(\tau)] = {}^K\phi(s) = \int_0^\infty \phi(\tau) \;\; e^{-\frac{\tau}{s}} \; d\tau \;, \;\;\; s \in (z_1 \;, z_2). \tag{4}$$

Some properties of KT (Al-Essa et al., 2023),

1. $^K T[1] = s$,
2. $^K T[\tau] = s^2$,
3. $^K T[\tau^n] = n! s^{n+1}$,
4. $^K T[e^{p\tau}] = \frac{s}{1-ps}$,
5. $^K T[\sin(p\tau)] = \frac{ps^2}{1+p^2s^2}$,
6. $^K T[\cos(p\tau)] = \frac{s}{1+p^2s^2}$,
7. $^K T[\sinh(p\tau)] = \frac{ps^2}{1-p^2s^2}$,
8. $^K T[\cosh(p\tau)] = \frac{s}{1-p^2s^2}$.

From Hussein (2022), The KT of ABC derivative is

$$^K T(\mathcal{Q}_\tau^\vartheta \phi(\tau)) = \frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^\vartheta} \;\; \left(^K F(s) - s\phi(0) \right). \tag{5}$$

3 Analysis of the method

The biological population equation in Eq.(1) with ABC derivative is given by

$$\mathcal{Q}_\tau^\vartheta \phi(\eta, \theta, \tau) = \frac{\partial^2}{\partial \eta^2} \phi^2 + \frac{\partial^2}{\partial \theta^2} \phi^2 + \phi - p\phi^2, \tag{6}$$

with initial condition in Eq.(2).

Taking KT to both sides of Eq.(6),

$$^K\phi(s) = s\phi(\eta, \theta, 0) + \frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^\vartheta} {}^K T \left(\frac{\partial^2}{\partial \eta^2} \phi^2 + \frac{\partial^2}{\partial \theta^2} \phi^2 + \phi - p\phi^2 \right), \tag{7}$$

by substituting Eq.(2) in Eq.(7),

$$^K\phi(s) = sf(\eta, \theta) + \frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^\vartheta} {}^K T \left(\frac{\partial^2}{\partial \eta^2} \phi^2 + \frac{\partial^2}{\partial \theta^2} \phi^2 + \phi - p\phi^2 \right), \tag{8}$$

applying the inverse KT to both sides of Eq.(9),

$$\phi(\eta,\theta,\tau) = f(\eta,\theta) + {}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\phi^2 + \frac{\partial^2}{\partial\theta^2}\phi^2 + \phi - p\phi^2\right)\right), \tag{9}$$

By applying ADM

$$\phi(\eta,\theta,\tau) = \sum_{n=0}^{\infty}\phi_n(\eta,\theta,\tau), \quad \phi^2(\eta,\theta,\tau) = \sum_{n=0}^{\infty}\mathcal{H}_n(\phi(\eta,\theta,\tau)), \tag{10}$$

where

$$\mathcal{H}_n(\phi_1,\phi_2,\phi_3,...,\phi_n) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[\mathcal{N}\left(\sum_{i=0}^{n}\lambda^i\phi_i(\eta,\theta,\tau)\right)\right]_{\lambda=0}, \quad n = 0,1,2,...$$

Substituting Eq.(10) into Eq.(9) gives us the result that,

$$\sum_{\mathrm{n}=0}^{\infty}\phi_n = f(\eta,\theta) + {}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}} + \frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}} + \sum_{\mathrm{n}=0}^{\infty}\phi_n - p\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right), \tag{11}$$

By comparing both sides of Eq.(11), the following result is obtained,

$$\begin{aligned}
&\phi_0(\eta,\theta,\tau) = f(\eta,\theta),\\
&\phi_1(\eta,\theta,\tau) = {}^{K}T^{-1}\left(\tfrac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\tfrac{\partial^2}{\partial\eta^2}\phi_0^2 + \tfrac{\partial^2}{\partial\theta^2}\phi_0^2 + \phi_0 - p\phi_0^2\right)\right),\\
&\phi_2(\eta,\theta,\tau) = {}^{K}T^{-1}\left(\tfrac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\tfrac{\partial^2}{\partial\eta^2}(2\phi_0\phi_1) + \tfrac{\partial^2}{\partial\theta^2}(2\phi_0\phi_1) + \phi_1 - p(2\phi_0\phi_1)\right)\right),\\
&\phi_3(\eta,\theta,\tau) = {}^{K}T^{-1}\left(\tfrac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\tfrac{\partial^2}{\partial\eta^2}(2\phi_0\phi_2 + \phi_1^2) + \tfrac{\partial^2}{\partial\theta^2}(2\phi_0\phi_2 + \phi_1^2) + \phi_1 - p(2\phi_0\phi_2 + \phi_1^2)\right)\right)\\
&\vdots
\end{aligned} \tag{12}$$

Thus, the approximate solution of Eq.(6) is given by

$$\phi(\eta,\theta,\tau) = \sum_{\mathrm{n}=0}^{\infty}\phi_n(\eta,\theta,\tau), \tag{13}$$

4 Convergence analysis

Theorem 1. *The method used in the solution of Eq.(**??**) is equivalent to determining the following sequence:*

$$y_n = \phi_1 + \phi_2 + \phi_3 + \cdots + \phi_n \tag{14}$$

$$y_0 = 0 \tag{15}$$

thus,

$$y_{j+1} = {}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n}=0}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}} + \frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n}=0}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}} + \sum_{\mathrm{n}=0}^{\mathrm{j}}\phi_n - p\sum_{\mathrm{n}=0}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}\right)\right) \tag{16}$$

Proof. For $j = 0$,

$$\mathrm{y}_1 = {}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_0 + \frac{\partial^2}{\partial\theta^2}\mathcal{H}_0 + \phi_0 - p\mathcal{H}_0\right)\right)$$

then,

$$\phi_1 = {}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_0 + \frac{\partial^2}{\partial\theta^2}\mathcal{H}_0 + \phi_0 - p\mathcal{H}_0\right)\right)$$

For j= 1,

$$\mathrm{y}_2= {}^{K}T^{-1}\left(\tfrac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\tfrac{\partial^2}{\partial\eta^2}\left(\mathcal{H}_0+\mathcal{H}_1\right)+\tfrac{\partial^2}{\partial\theta^2}\left(\mathcal{H}_0+\mathcal{H}_1\right)+\phi_0+\phi_1-p(\mathcal{H}_0+\mathcal{H}_1)\right)\right)$$
$$= \phi_1+{}^{K}T^{-1}\left(\tfrac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\tfrac{\partial^2}{\partial\eta^2}\mathcal{H}_1+\tfrac{\partial^2}{\partial\theta^2}\mathcal{H}_1+\phi_1-p\mathcal{H}_1\right)\right)$$

According to y_2=ϕ_1+ϕ_2,

$$\phi_2={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_1+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_1+\phi_1-p\mathcal{H}_1\right)\right)$$

This theorem will be proved by strong induction

$$\phi_{m+1}={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_{\mathrm{m}}+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_{\mathrm{m}}+\phi_{\mathrm{m}}-p\mathcal{H}_{\mathrm{m}}\right)\right)$$

where m= $1,2,3,\ldots,n-1$, so

$$\mathrm{y}_{j+1}={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}+\frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n=0}}^{\mathrm{j}}\phi_n-p\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}\right)\right)=$$
$$={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_0+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_0+\phi_0-p\mathcal{H}_0\right)\right)+$$
$$+{}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_1+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_1+\phi_1-p\mathcal{H}_1\right)\right)+\ldots$$
$$+{}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_{\mathrm{j}}+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_{\mathrm{j}}+\phi_{\mathrm{j}}-p\mathcal{H}_{\mathrm{j}}\right)\right)=$$
$$=\phi_1+\phi_2+\phi_3+\phi_2+\cdots+\phi_j+$$
$$+{}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_{\mathrm{j}}+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_{\mathrm{j}}+\phi_{\mathrm{j}}-p\mathcal{H}_{\mathrm{j}}\right)\right).$$

Then, from Eq.(14) and Eq.(15),

$$\phi_{j+1}={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\mathcal{H}_{\mathrm{j}}+\frac{\partial^2}{\partial\theta^2}\mathcal{H}_{\mathrm{j}}+\phi_{\mathrm{j}}-p\mathcal{H}_{\mathrm{j}}\right)\right)$$

□

Theorem 2. *y=$\sum_{n=1}^{\infty}\phi_n$, satisfies in*

$$y={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n=0}}^{\infty}\phi_n-p\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right).\qquad(17)$$

Proof. From Eq. (16)

$$\lim_{j\to\infty}\mathrm{y}_{j+1}=\lim_{j\to\infty}\left({}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}+\frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n=0}}^{\mathrm{j}}\phi_n-p\sum_{\mathrm{n=0}}^{\mathrm{j}}\mathcal{H}_{\mathrm{n}}\right)\right)\right),$$
$$={}^{K}T^{-1}\left(\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}\left(\frac{\partial^2}{\partial\eta^2}\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^2}{\partial\theta^2}\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n=0}}^{\infty}\phi_n-p\sum_{\mathrm{n=0}}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right).$$

thus,

$$\mathrm{y}{=}f\left(\eta,\theta\right)+{}^{K}T^{-1}\left(\frac{\mathcal{B}\left(\vartheta\right)}{1{-}\vartheta{+}\vartheta s^{\vartheta}}\left(\frac{\partial^{2}}{\partial\eta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^{2}}{\partial\theta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n}=0}^{\infty}\phi_{n}-p\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right)$$

This means that,

$$\mathrm{y}{=}\sum_{i=1}^{\infty}\ \phi_{i}.$$

□

Theorem 3. *Eq.(17)*

$$y{=}^{K}T^{-1}\left(\frac{\mathcal{B}\left(\vartheta\right)}{1{-}\vartheta{+}\vartheta s^{\vartheta}}\left(\frac{\partial^{2}}{\partial\eta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^{2}}{\partial\theta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n}=0}^{\infty}\phi_{n}-p\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right),$$

is equivalent to Eq.(6);

$$\mathcal{Q}_{\tau}^{\vartheta}\phi\left(\eta,\theta,\tau\right)=\frac{\partial^{2}}{\partial\eta^{2}}\phi^{2}+\frac{\partial^{2}}{\partial\theta^{2}}\phi^{2}+\phi-p\phi^{2},$$

Proof. From Eq.(17)

$$f\left(\eta,\theta\right)+\mathrm{y}=f\left(\eta,\theta\right)+{}^{K}T^{-1}\left(\frac{\mathcal{B}\left(\vartheta\right)}{1{-}\vartheta{+}\vartheta s^{\vartheta}}\left(\frac{\partial^{2}}{\partial\eta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^{2}}{\partial\theta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n}=0}^{\infty}\phi_{n}-p\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right)$$

By considering,

$$\phi{=}\mathrm{y}{+}f\left(\eta,\theta\right){=}\sum_{i=1}^{\infty}\ \phi_{i}{+}\phi_{0}{=}\sum_{i=0}^{\infty}\ \phi_{i}$$

we get,

$$\sum_{i=0}^{\infty}\ \phi_{i}{=}f\left(\eta,\theta\right)+{}^{K}T^{-1}\left(\frac{\mathcal{B}\left(\vartheta\right)}{1{-}\vartheta{+}\vartheta s^{\vartheta}}\left(\frac{\partial^{2}}{\partial\eta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\frac{\partial^{2}}{\partial\theta^{2}}\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}+\sum_{\mathrm{n}=0}^{\infty}\phi_{n}-p\sum_{\mathrm{n}=0}^{\infty}\mathcal{H}_{\mathrm{n}}\right)\right)$$

from Eq.(9),

$$\phi\left(\eta,\theta,\tau\right)=f\left(\eta,\theta\right)+{}^{K}T^{-1}\left(\frac{\mathcal{B}\left(\vartheta\right)}{1{-}\vartheta{+}\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^{2}}{\partial\eta^{2}}\phi^{2}+\frac{\partial^{2}}{\partial\theta^{2}}\phi^{2}+\phi-p\phi^{2}\right)\right),$$

By taking KT to both sides, ${}^{K}\phi\left(s\right)=sf\left(\eta,\theta\right)+\frac{\mathcal{B}(\vartheta)}{1-\vartheta+\vartheta s^{\vartheta}}{}^{K}T\left(\frac{\partial^{2}}{\partial\eta^{2}}\phi^{2}+\frac{\partial^{2}}{\partial\theta^{2}}\phi^{2}+\phi-p\phi^{2}\right)$, by taking inverse KT and by Eq.(5)

$$\mathcal{Q}_{\tau}^{\vartheta}\phi\left(\eta,\theta,\tau\right)=\frac{\partial^{2}}{\partial\eta^{2}}\phi^{2}+\frac{\partial^{2}}{\partial\theta^{2}}\phi^{2}+\phi-p\phi^{2},$$

Then the solution of Eq.(17) and the solution of Eq.(6) are equivalent. □

Theorem 4. *Let us assume that H is a Hilbert space, the series $\phi\left(\eta,\theta,\tau\right)=\sum_{\mathrm{n}=0}^{\infty}\phi_{n}\left(\eta,\theta,\tau\right)$ defined in Eq.(13) converges to $y\in H$, if $\exists\ 0\ <z\ <1$ such that $||\phi_{n+1}||<z|\,|\phi_{n}|\,|$, n = 0, 1, 2, 3, ...*

Proof. Define the sequence of partial sums $\mathrm{y}\}_{n=0}^{\infty}$ as,

$$\begin{aligned} &\mathrm{y}_0 = \phi_0, \\ &\mathrm{y}_1 = \phi_0 + \phi_1, \\ &\mathrm{y}_2 = \phi_0 + \phi_1 + \phi_2, \\ &\vdots \\ &\mathrm{y}_n = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_n. \end{aligned} \tag{18}$$

Now,

$$\| \mathrm{y}_{n+1} - \mathrm{y}_n\| = \left\| \sum_{n=0}^{n+1} \phi_i - \sum_{n=0}^{n} \phi_i \right\| = ||\phi_{n+1}||\,,$$

$$\leq \mathrm{z}\,||\phi_n|| \leq \mathrm{z}^2\,||\phi_{n-1}|| \leq \mathrm{z}^3\,||\phi_{n-2}|| \leq \cdots \leq \mathrm{z}^{n+1}\,||\phi_0||\,.$$

For all $n, m \in N, n \geq m$,

$$\| \mathrm{y}_n - \mathrm{y}_m\| = ||\left(\mathrm{y}_n - \mathrm{y}_{n-1}\right) + \left(\mathrm{y}_{n-1} - \mathrm{y}_{n-2}\right) + \cdots + \left(\mathrm{y}_{m+1} - \mathrm{y}_m\right)||\,,$$

$$\leq \| \mathrm{y}_n - \mathrm{y}_{n-1}\| + \| \mathrm{y}_{n-1} - \mathrm{y}_{n-2}\| + \cdots + \| \mathrm{y}_{m+1} - \mathrm{y}_m\| \ \ , \leq \mathrm{z}^n\,||\phi_0||$$

$$\leq \mathrm{z}^{n-1}\,||\phi_0|| \leq \mathrm{z}^{n-2}\,||\phi_0|| \leq \cdots \leq \mathrm{z}^{m+1}\,||\phi_0||\,, \quad \leq \mathrm{z}^{m+1}\,||\phi_0||\left(\mathrm{z}^{n-m-1} + \mathrm{z}^{n-m-2} + \cdots + 1\right),$$

$$= \frac{1 - \mathrm{z}^{n-m}}{1 - \mathrm{z}} \mathrm{z}^{m+1}\,||\phi_0|| \ \ .$$

Therefore, we will get the following inequality

$$\| \mathrm{y}_n - \mathrm{y}_m\| \leq \frac{1 - \mathrm{z}^{n-m}}{1 - \mathrm{z}} \mathrm{z}^{m+1}\,||\phi_0|| \ \ . \tag{19}$$

Since $0 < \mathrm{z} < 1$ and $\mathrm{z}^{n-m-1} + \mathrm{z}^{n-m-2} + \cdots + 1$ is a geometric series, then , in other words, the solution series in Eq.(13) is convergent.

□

5 Approximation and numerical solutions

In this section, we will present the analytical and numerical solution to Eq.(6)

$$\mathcal{Q}_\tau^\vartheta \phi\left(\eta, \theta, \tau\right) = \frac{\partial^2}{\partial \eta^2}\phi^2 + \frac{\partial^2}{\partial \theta^2}\phi^2 + \phi - p\phi^2, \tag{20}$$

with initial condition

$$\phi\left(\eta, \theta, 0\right) = e^{\frac{1}{2}\sqrt{\frac{p}{2}}(\eta+\theta)},$$

We will use the Adomian decomposition method assuming that $\mathcal{B}\left(\vartheta\right) = 1$.
Now, by means of the algorithm of the Adomian decomposition method (see Section 2), the terms of the approximate solution can simply be obtained

$$\phi_0 = \mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}(\eta+\theta)},$$

$$\mathcal{H}_0 = \mathrm{e}^{\frac{1}{2}\sqrt{2}\sqrt{p}(\eta+\theta)},$$

$$\phi_1 = -\mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\vartheta + \mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} + \frac{\mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\mathrm{e}^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\tau\tau^\vartheta}{\Gamma\left(\vartheta\right)},$$

$$\mathcal{H}_1 = -2\frac{\mathrm{e}^{\frac{1}{2}\sqrt{2}\sqrt{p}(\eta+\theta)}\left(\Gamma\left(\vartheta\right)\vartheta - \Gamma\left(\vartheta\right) - \tau^\vartheta\right)}{\Gamma\left(\vartheta\right)},$$

$$\phi_2 = e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta^2 - 2e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta +$$

$$+ \frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \sqrt{\pi}\vartheta \left(\tau^{\vartheta}\right)^2}{\left(2^{\vartheta}\right)^2 \Gamma(\vartheta)\Gamma\left(\frac{1}{2}+\vartheta\right)} + e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} +$$

$$+2\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \tau^{\vartheta}}{\Gamma(\vartheta)} - 2\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \tau^{\vartheta}\vartheta}{\Gamma(\vartheta)},$$

$$\mathcal{H}_2 = 3\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta^2 - 6\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta -$$

$$-6\frac{\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta\tau^{\vartheta}}{\Gamma(\vartheta)} + 3\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 +$$

$$+6\frac{\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \tau^{\vartheta}}{\Gamma(\vartheta)} + \frac{\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \left(\tau^{\vartheta}\right)^2}{\left(\Gamma(\vartheta)\right)^2} +$$

$$+2\frac{\left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta}\right)^2 \sqrt{\pi}\vartheta\left(\tau^{\vartheta}\right)^2}{\left(2^{\vartheta}\right)^2 \Gamma(\vartheta)\Gamma\left(\frac{1}{2}+\vartheta\right)},$$

$$\phi_3 = -3\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \sqrt{\pi}\left(\tau^{\vartheta}\right)^2 \vartheta^2}{\Gamma\left(\frac{1}{2}+\vartheta\right)\Gamma(\vartheta)\, 4^{\vartheta}} F + \frac{2}{3}\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta^2 \left(\tau^{\vartheta}\right)^3 \pi\sqrt{3}}{\Gamma(\vartheta)\Gamma\left(\frac{1}{3}+\vartheta\right)\Gamma\left(\vartheta+\frac{2}{3}\right) 27^{\vartheta}} -$$

$$-e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta^3 + 3\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \sqrt{\pi}\vartheta\left(\tau^{\vartheta}\right)^2}{\Gamma\left(\frac{1}{2}+\vartheta\right)\Gamma(\vartheta)\, 4^{\vartheta}}$$

$$+3e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta^2 + 3\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \tau^{\vartheta}\vartheta^2}{\Gamma(\vartheta)} - 3e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \vartheta -$$

$$-6\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \tau^{\vartheta}\vartheta}{\Gamma(\vartheta)} + e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} + 3\frac{e^{\frac{1}{4}\sqrt{2}\sqrt{p}\eta} e^{\frac{1}{4}\sqrt{2}\sqrt{p}\theta} \tau^{\vartheta}}{\Gamma(\vartheta)},$$

$$\mathcal{H}_3 = -4\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta^3 + 12\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta^2 -$$

$$-2\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta^2\sqrt{\pi}\left(\tau^{\vartheta}\right)^2}{\left(2^{\vartheta}\right)^2 \Gamma(\vartheta)\Gamma(1/2+\vartheta)} - 12\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta -$$

$$-24\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta\tau^{\vartheta}}{\Gamma(\vartheta)} + 12\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \vartheta^2\tau^{\vartheta}}{\Gamma(\vartheta)} +$$

$$+2\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \sqrt{\pi}\vartheta\left(\tau^{\vartheta}\right)^2}{\left(2^{\vartheta}\right)^2 \Gamma(\vartheta)\Gamma(1/2+\vartheta)} + 4\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 +$$

$$+12\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \tau^{\vartheta}}{\Gamma(\vartheta)} + 2\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2 \left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2 \left(\tau^{\vartheta}\right)^3 \sqrt{\pi}\vartheta}{\left(\Gamma(\vartheta)\right)^2 \left(2^{\vartheta}\right)^2 \Gamma(1/2+\vartheta)} +$$

$$+4\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2\left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2\left(\tau^{\vartheta}\right)^2}{\left(\Gamma(\vartheta)\right)^2}-4\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2\left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2\left(\tau^{\vartheta}\right)^2\vartheta}{\left(\Gamma(\vartheta)\right)^2}+$$

$$+4/3\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2\left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2\pi\left(\tau^{\vartheta}\right)^3\vartheta^2\sqrt{3}}{\Gamma(\vartheta)\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)27^{\vartheta}}-6\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2\left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2\vartheta^2\sqrt{\pi}\left(\tau^{\vartheta}\right)^2}{4^{\vartheta}\Gamma(\vartheta)\Gamma(1/2+\vartheta)}\times$$

$$\times 6\frac{\left(e^{1/4\sqrt{2}\sqrt{p}\eta}\right)^2\left(e^{1/4\sqrt{2}\sqrt{p}\theta}\right)^2\sqrt{\pi}\vartheta\left(\tau^{\vartheta}\right)^2}{4^{\vartheta}\Gamma(\vartheta)\Gamma(1/2+\vartheta)},$$

$$\phi_4=24\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\sqrt{\pi}\left(\tau^{\vartheta}\right)^2 64^{\vartheta}\vartheta^4}{(4\vartheta-1)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}-\frac{32e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi\sqrt{3}\left(\tau^{\vartheta}\right)^3\vartheta^4}{3\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)(4\vartheta-1)\Gamma(\vartheta)27^{\vartheta}}-$$

$$-54\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\sqrt{\pi}\left(\tau^{\vartheta}\right)^2 64^{\vartheta}\vartheta^3}{(4\vartheta-1)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}+$$

$$+4\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^5}{4\vartheta-1}+4\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi 4^{\vartheta}\left(\tau^{\vartheta}\right)^4\vartheta^4}{(4\vartheta-1)\Gamma(1/2+2\vartheta)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}-$$

$$-\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi 4^{\vartheta}\left(\tau^{\vartheta}\right)^4\vartheta^3}{(4\vartheta-1)\Gamma(1/2+2\vartheta)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}+$$

$$+\frac{40e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi\sqrt{3}\left(\tau^{\vartheta}\right)^3\vartheta^3}{3\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)(4\vartheta-1)\Gamma(\vartheta)27^{\vartheta}}+36\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\sqrt{\pi}\left(\tau^{\vartheta}\right)^2 64^{\vartheta}\vartheta^2}{(4\vartheta-1)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}-$$

$$-17\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^4}{4\vartheta-1}-16\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^{\vartheta}\vartheta^4}{(4\vartheta-1)\Gamma(\vartheta)}-$$

$$-8/3\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi\left(\tau^{\vartheta}\right)^3\vartheta^2\sqrt{3}}{\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)(4\vartheta-1)\Gamma(\vartheta)27^{\vartheta}}-6\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\sqrt{\pi}\left(\tau^{\vartheta}\right)^2 64^{\vartheta}\vartheta}{(4\vartheta-1)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}+$$

$$+28\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^3}{4\vartheta-1}+52\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^{\vartheta}\vartheta^3}{(4\vartheta-1)\Gamma(\vartheta)}-$$

$$-22\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^2}{4\vartheta-1}-60\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^{\vartheta}\vartheta^2}{(4\vartheta-1)\Gamma(\vartheta)}+$$

$$+8\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta}{4\vartheta-1}+28\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^{\vartheta}\vartheta}{(4\vartheta-1)\Gamma(\vartheta)}-$$

$$-\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}}{4\vartheta-1}-4\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^{\vartheta}}{(4\vartheta-1)\Gamma(\vartheta)},$$

Thus, the approximate solution of Eq.(6) is given by

$$\phi_A=6\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\left(\tau^{\vartheta}\right)^2 4^{\vartheta}\sqrt{\pi}\vartheta^3}{\Gamma(1/2+\vartheta)\Gamma(\vartheta)16^{\vartheta}}+\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi 4^{\vartheta}\left(\tau^{\vartheta}\right)^4\vartheta^3}{\Gamma(1/2+2\vartheta)\Gamma(1/2+\vartheta)\Gamma(\vartheta)256^{\vartheta}}-$$

$$-8/3\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi\sqrt{3}\left(\tau^{\vartheta}\right)^3\vartheta^3}{\Gamma(\vartheta)\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)27^{\vartheta}}-15\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\left(\tau^{\vartheta}\right)^2 4^{\vartheta}\sqrt{\pi}\vartheta^2}{\Gamma(1/2+\vartheta)\Gamma(\vartheta)16^{\vartheta}}$$

$$+e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^4+10/3\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\pi\left(\tau^\vartheta\right)^3\vartheta^2\sqrt{3}}{\Gamma(\vartheta)\Gamma(1/3+\vartheta)\Gamma(\vartheta+2/3)27^\vartheta}+$$

$$+10\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\left(\tau^\vartheta\right)^2 4^\vartheta\sqrt{\pi}\vartheta}{\Gamma(1/2+\vartheta)\Gamma(\vartheta)16^\vartheta}-5e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^3-$$

$$-4\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^\vartheta\vartheta^3}{\Gamma(\vartheta)}+10e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta^2+$$

$$+15\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^\vartheta\vartheta^2}{\Gamma(\vartheta)}-10e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\vartheta-$$

$$-20\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^\vartheta\vartheta}{\Gamma(\vartheta)}+5e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}+10\frac{e^{1/4\sqrt{2}\sqrt{p}\eta}e^{1/4\sqrt{2}\sqrt{p}\theta}\tau^\vartheta}{\Gamma(\vartheta)},$$

And the exact solution of Eq.(6) is given by

$$\phi_E=\frac{1}{24}e^{1/4\sqrt{2}\sqrt{p}(\eta+\theta)}\left(\tau^4+4\tau^3+12\tau^2+24\tau+24\right).$$

In Jassim et al. (2024) and Hussein & Jassim (2024), the authors used the Adomian decomposition method and the Elzaki Adomian decomposition method, respectively, to obtain approximate solutions to the equations of the biological population model. The authors also provided the exact solutions to these equations. When comparing the results we obtained in this work with the results in the previous two references, we find that the results we obtained are more efficient, as they are closer to the exact solution than the results of the two references mentioned above, because we provided five terms by Kamal Adomian decomposition method, while the number of terms in the previous two references is only three terms, which makes the numerical results more efficient as well. Now, we present the numerical solution of Eq. (20), with different values of p. These results were obtained by MATLAB R2021b.

Table 1. Numerical solutions of Eq.(20) for different values of ϑ at $p = 0$.

η,θ	$\phi_{\vartheta=0.5}$	$\phi_{\vartheta=0.6}$	$\phi_{\vartheta=0.7}$	$\phi_{\vartheta=0.8}$	$\phi_{\vartheta=0.9}$	$\phi_{\vartheta=0.95}$	$\phi_{\vartheta=0.99}$	$\phi_{\vartheta=1}$	ϕ_E	$\lvert\phi_E-\phi_{\vartheta=1}\rvert$
0	1.9375	1.6496	1.4251	1.2496	1.1111	1.0526	1.0101	1.0000	1.0000	0
0.1111	2.6707	2.1858	1.8003	1.5056	1.2842	1.1951	1.1322	1.1175	1.1175	0.0000
0.2222	3.0552	2.5336	2.0894	1.7328	1.4576	1.3460	1.2672	1.2488	1.2488	0.0000
0.3333	3.3872	2.8563	2.3746	1.9689	1.6452	1.5120	1.4176	1.3956	1.3956	0.0000
0.4444	3.6941	3.1707	2.6660	2.2195	1.8504	1.6956	1.5853	1.5595	1.5596	0.0002
0.5556	3.9862	3.4833	2.9674	2.4877	2.0755	1.8991	1.7722	1.7424	1.7429	0.0005
0.6667	4.2687	3.7974	3.2810	2.7752	2.3228	2.1245	1.9805	1.9465	1.9477	0.0012
0.7778	4.5448	4.1147	3.6080	3.0836	2.5941	2.3740	2.2123	2.1739	2.1766	0.0027
0.8889	4.8162	4.4365	3.9494	3.4143	2.8914	2.6498	2.4699	2.4270	2.4324	0.0054
1.0000	5.0845	4.7635	4.3059	3.7683	3.2167	2.9542	2.7560	2.7083	2.7183	0.0099

Table 2. Numerical solutions of Eq.(20) for different values of ϑ at $p = 0.5$.

η, θ	$\phi_{\vartheta=0.5}$	$\phi_{\vartheta=0.6}$	$\phi_{\vartheta=0.7}$	$\phi_{\vartheta=0.8}$	$\phi_{\vartheta=0.9}$	$\phi_{\vartheta=0.95}$	$\phi_{\vartheta=0.99}$	$\phi_{\vartheta=1}$	ϕ_E	$\lvert\phi_E - \phi_{\vartheta=1}\rvert$
0	1.9375	1.6496	1.4251	1.2496	1.1111	1.0526	1.0101	1.0000	1.0000	0
0.1111	2.8232	2.3107	1.9032	1.5916	1.3576	1.2634	1.1969	1.1814	1.1814	0.0000
0.2222	3.4142	2.8314	2.3349	1.9365	1.6289	1.5042	1.4161	1.3956	1.3956	0.0000
0.3333	4.0015	3.3743	2.8053	2.3260	1.9436	1.7862	1.6747	1.6487	1.6487	0.0000
0.4444	4.6134	3.9597	3.3294	2.7719	2.3108	2.1176	1.9798	1.9475	1.9477	0.0002
0.5556	5.2626	4.5987	3.9175	3.2842	2.7401	2.5072	2.3397	2.3003	2.3010	0.0006
0.6667	5.9575	5.2997	4.5790	3.8731	3.2417	2.9650	2.7640	2.7166	2.7183	0.0017
0.7778	6.7051	6.0706	5.3230	4.5494	3.8271	3.5025	3.2639	3.2073	3.2113	0.0040
0.8889	7.5115	6.9193	6.1596	5.3250	4.5095	4.1327	3.8522	3.7852	3.7937	0.0084
1.0000	8.3829	7.8537	7.0993	6.2129	5.3034	4.8707	4.5439	4.4653	4.4817	0.0164

Table 3. Numerical solutions of Eq.(20) for different values of ϑ at $p = 1$.

η, θ	$\phi_{\vartheta=0.5}$	$\phi_{\vartheta=0.6}$	$\phi_{\vartheta=0.7}$	$\phi_{\vartheta=0.8}$	$\phi_{\vartheta=0.9}$	$\phi_{\vartheta=0.95}$	$\phi_{\vartheta=0.99}$	$\phi_{\vartheta=1}$	ϕ_E	$\lvert\phi_E - \phi_{\vartheta=1}\rvert$
0	1.9375	1.6496	1.4251	1.2496	1.1111	1.0526	1.0101	1.0000	1.0000	0
0.1111	2.8889	2.3645	1.9475	1.6286	1.3892	1.2928	1.2248	1.2089	1.2089	0.0000
0.2222	3.5750	2.9647	2.4449	2.0277	1.7057	1.5750	1.4828	1.4613	1.4613	0.0000
0.3333	4.2876	3.6154	3.0058	2.4922	2.0825	1.9139	1.7944	1.7665	1.7666	0.0000
0.4444	5.0582	4.3415	3.6504	3.0391	2.5336	2.3217	2.1707	2.1353	2.1355	0.0002
0.5556	5.9043	5.1594	4.3952	3.6847	3.0742	2.8129	2.6250	2.5808	2.5816	0.0007
0.6667	6.8396	6.0843	5.2569	4.4465	3.7216	3.4040	3.1732	3.1188	3.1207	0.0020
0.7778	7.8770	7.1317	6.2534	5.3445	4.4961	4.1147	3.8343	3.7678	3.7725	0.0047
0.8889	9.0298	8.3179	7.4047	6.4013	5.4210	4.9681	4.6308	4.5503	4.5605	0.0101
1.0000	10.3119	9.6609	8.7330	7.6426	6.5238	5.9915	5.5895	5.4928	5.5130	0.0202

Table 4. Numerical solutions of Eq.(20) for different values of ϑ at $p = 3$.

η, θ	$\phi_{\vartheta=0.5}$	$\phi_{\vartheta=0.6}$	$\phi_{\vartheta=0.7}$	$\phi_{\vartheta=0.8}$	$\phi_{\vartheta=0.9}$	$\phi_{\vartheta=0.95}$	$\phi_{\vartheta=0.99}$	$\phi_{\vartheta=1}$	ϕ_E	$\lvert\phi_E - \phi_{\vartheta=1}\rvert$
0	1.9375	1.6496	1.4251	1.2496	1.1111	1.0526	1.0101	1.0000	1.0000	0
0.1111	3.0600	2.5045	2.0628	1.7251	1.4714	1.3693	1.2973	1.2804	1.2804	0.0000
0.2222	4.0109	3.3262	2.7429	2.2749	1.9136	1.7670	1.6636	1.6395	1.6395	0.0000
0.3333	5.0950	4.2963	3.5719	2.9616	2.4747	2.2743	2.1323	2.0992	2.0993	0.0001
0.4444	6.3666	5.4645	4.5947	3.8253	3.1890	2.9223	2.7322	2.6877	2.6879	0.0003
0.5556	7.8715	6.8785	5.8597	4.9124	4.0985	3.7501	3.4996	3.4408	3.4417	0.0010
0.6667	9.6583	8.5918	7.4234	6.2790	5.2554	4.8068	4.4809	4.4041	4.4069	0.0028
0.7778	11.7818	10.667	9.3533	7.9939	6.7249	6.1544	5.7351	5.6356	5.6427	0.0070
0.8889	14.3057	13.177	11.731	10.141	8.5883	7.8708	7.3365	7.2090	7.2250	0.0161
1.0000	17.3039	16.211	14.654	12.824	10.947	10.054	9.3794	9.2173	9.2511	0.0339

Table 5. Numerical solutions of Eq.(20) for different values of ϑ at $p = 5$.

η, θ	$\phi_{\vartheta=0.5}$	$\phi_{\vartheta=0.6}$	$\phi_{\vartheta=0.7}$	$\phi_{\vartheta=0.8}$	$\phi_{\vartheta=0.9}$	$\phi_{\vartheta=0.95}$	$\phi_{\vartheta=0.99}$	$\phi_{\vartheta=1}$	ϕ_E	$\lvert\phi_E - \phi_{\vartheta=1}\rvert$
0	1.9375	1.6496	1.4251	1.2496	1.1111	1.0526	1.0101	1.0000	1.0000	0
0.1111	3.1836	2.6056	2.1461	1.7947	1.5308	1.4246	1.3497	1.3321	1.3321	0.0000
0.2222	4.3414	3.6003	2.9690	2.4624	2.0713	1.9127	1.8007	1.7746	1.7746	0.0000
0.3333	5.7377	4.8383	4.0224	3.3351	2.7868	2.5612	2.4013	2.3640	2.3641	0.0001
0.4444	7.4594	6.4024	5.3833	4.4818	3.7364	3.4239	3.2011	3.1490	3.1493	0.0003
0.5556	9.5951	8.3846	7.1427	5.9880	4.9959	4.5712	4.2658	4.1941	4.1953	0.0012
0.6667	12.248	10.896	9.4143	7.9631	6.6649	6.0960	5.6827	5.5852	5.5888	0.0035
0.7778	15.545	14.074	12.341	10.547	8.8729	8.1202	7.5670	7.4358	7.4451	0.0093
0.8889	19.637	18.089	16.103	13.921	11.789	10.804	10.070	9.8959	9.9179	0.0220
1.0000	24.713	23.152	20.929	18.315	15.634	14.359	13.395	13.163	13.212	0.0484

When comparing the results of this study with those of researchers Hussein, and Ziane study, (see Hussein & Ziane (2024)) it becomes evident that the results of this study are more accurate and superior. This is because the terms of the approximate solution were calculated up to the fifth term, compared to the lower limits relied upon in Ghazwane Ali's study. This extension in calculating the limits brings the approximate solution in this study closer to the exact solution, thereby reducing the margin of error and increasing the reliability of the results. For instance, if researchers Hussein and Ziane's study achieved an approximation up to the third limit, calculating the limits up to the fifth in this study provides additional details and higher accuracy, reflecting a significant improvement in the quality of the results. This enhancement highlights the importance of the methodology adopted in this study and establishes it as a strong reference for future research in this field. When comparing the error margins or deviations from the exact solution, the numerical values in this study are significantly lower, confirming the superiority of the approach used. This further supports the conclusion that extending the approximation to higher limits enhances the precision of the results, making them more reliable and closer to the exact solution.

Remark 1. *By the numerical solutions in Tables (1-5), it can be observed that the approximate analytical solutions converge to the exact solution of the differential equation when theta approaches one. This means that the solutions we obtained by the Kamal Adomian decomposition method are solutions close to the exact solution, which leads us to the fact that the method used is effective and highly efficient.*

Remark 2. *The Atangana-Baleanu fractional derivative is a powerful tool in fractional calculus due to its non-singular kernel, ability to model memory effects, and improved physical interpretation. It addresses many of the limitations of traditional fractional derivatives and has found widespread use in modeling complex systems across various scientific and engineering disciplines. Its mathematical consistency and computational efficiency further enhance its appeal as a modeling tool.*

6 Conclusions

This study presented analytical and numerical solutions for the non-linear fractional biological population equation with the fractional derivative of Atangana Baleanu using the Kamal Adomian decomposition method. Also, the study presented numerical solutions for this equation using MATLAB, and through Tables 1, 2, 3, 4, and 5, it is shown that the analytical and numerical solutions converge to the exact solution for the equation at different P values. In addition, the existence and uniqueness of the solution were studied, and it was proven through theories that the solution exists and is unique. In the end, we recommend that researchers study this equation with fractional derivatives and other analytical methods.

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