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Ministry of Higher Education
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University of Basrah
College of Science
Department of Mathematics

Numerical Solutions of Variable Order Fractional Partial Differential Equations with Delay

A Thesis Submitted to the Council of the College of Science - University of Basrah as a Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

by

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Supervised by

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بِسْمِ اللهِ الرَّحْمَنِ الرَّحِيمِ

﴿اللهُ نُورُ السَّماوَاتِ وَالأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا مِصْبَاحٌ الْمِصْبَاحُ فِي زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ يُوقَدُ مِنْ شَجَرَةٍ مُبَارَكَةٍ زَيْتُونَةٍ لَا شَرْقِيَّةٍ وَلَا غَرْبِيَّةٍ يَكَادُ زَيْتُهَا يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ نُورٌ عَلَى نُورٍ يَهْدِي اللهُ لِنُورِهِ مَنْ يَشَاءُ وَيَضْرِبُ اللهُ الأَمْثَالَ لِلنَّاسِ وَاللهُ بِكُلِّ شَيْءٍ عَلِيمٌ﴾

صدق الله ألعلي العظيم

{الآية 35 من سورة النور}

Certification

I certify that the thesis entitled "***Numerical Solutions of Variable Order Fractional Partial Differential Equations with Delay*** " which is being submitted by ***Adnan Kalaf Farhood*** was prepared under my supervision at the Department of Mathematics, College of Science, University of Basrah in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Signature:......................
Prof.Dr. Osama H.Mohammed
Date: / 8 / 2023

In view of the available recommendations, I forward this thesis for debate by the examining committee.

Signature:........................
Assis Prof. Dr. Mohammed S. Kadhim
Head of the Math. Dept.
Date: / 8 / 2023

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Committee Report

We certify that we have read the thesis entitled "***Numerical Solutions of Variable Order Fractional Partial Differential Equations with Delay*** " and as examining committee, we have examined the student in its content and in our opinion, it is adequate with standing as a thesis for the degree of Doctor of Philosophy in Mathematics.

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Date: / 11/ 2023

Dedication

I dedicate this work to the spirit of my mother.

To the fountain of warmth, love, and tenderness, the light that illuminated my path, my affectionate father .

To those who have done so much for me, help me in every adversity, my dear brothers and sisters.

Also, I dedicate this work to my family: my wife and my dear children. Their presence in my life is a constant source of encouragement, love, and happiness.

and

To anyone who gave me help:
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Finally, I would like to express my deep thanks for all of my family, including my mother, brothers, wife, and children, for their support, care, patience, and help.

Adnan
October 2023

Declaration

The contents of this thesis are our original work except where a specific acknowledgment is given. This thesis has not been submitted in whole or in part to any other university. Certain aspects of this thesis have been published/ submitted in the following papers:

[**24**] Adnan K. Farhood and Osama H Mohammed, Bushra A. Taha. Mohammed, Solving Fractional Time-Delay Diffusion Equation with Variable-Order Derivative Based on Shifted Legendre-Laguerre Operational Matrices,Arabian Journal of Mathematics, vol. 17, pp. 1-11, Springer, 2023.

[**25**] Adnan K. Farhood and Osama H. Mohammed, Homotopy perturbation method for solving time-fractional nonlinear Variable-Order Delay Partial Differential Equations, Partial Differential Equations in Applied Mathematics, vol. 7, pp. 100513, Elsevier, 2023.

[**26**] Adnan K. Farhood and Osama H, Shifted Chebyshev operational matrices to solve the fractional time-delay diffusion equation, Partial Differential Equations in Applied Mathematics, vol. 8, pp. 100538, Elsevier, 2023.

[*] Adnan K. Farhood and Osama H, Solving Fractional Time-Delay Diffusion Equation with Variable-Order Derivative Based on Shifted Chebyshev-Laguerre Operational Matrices, Results in Control and Optimization, under review.

Abstract

The main aim of this thesis focuses around four objectives:
The first objective adopts a novel technique to numerical solution for fractional time-delay diffusion equation with variable-order derivative (VFDDEs). As a matter of fact, the variable order fractional derivative (VFD) that has been used is in the Caputo sense. The first step of this technique is constructive on the construction of the solution using the shifted Legendre-Laguerre polynomials with unknown coefficients. The second step involves using a combination of the collocation method and the operational matrices (OMs) of the shifted Legendre-Laguerre polynomials, as well as the Newton-Cotes nodal points, to find the unknown coefficients. The final step focuses on solving the resulting algebraic equations by employ Newton's iterative method. The efficiency and accuracy of the proposed technique will be investigated by some provided examples.
The second objective Homotopy perturbation method is extend to derive the approximate solution of the variable order fractional partial differential equations with time delay. The variable order fractional derivative is taken in the Caputo sense. By employing the Homotopy perturbation method the explicit approximate solutions are found. The error and convergence analysis of the Homotopy perturbation method has been discussed for the applicability of the method. The absolute errors and the approximate solutions are presented graphically and by tables at the values of various variable fractional order. From the results of the illustrated examples, we can Judge that the Homotopy perturbation method is very effective, and simple accelerates the rapid convergence of the solution.

The third objective proposed the, Chebyshev operational matrices collocation technique is proposed for finding the solution of variable order derivative fractional diffusion equation with proportional delay . The beginning of this approach is based on the construction of the solution using the shifted Chebyshev polynomials with unknown coefficients. After that, we performed the Newton-Cotes nodal points, the Chebyshev polynomials operational matrices, and the collocation method for calculating the unknown coefficients. According to the described technique, we get an algebraic system of nonlinear equations which can be solved easily by using Newton's iterative method. The efficiency and applicability of suggested approach are illustrated by some tested examples.
The fourth objective, a modified numerical method for the solution of the diffusion equation with variable fractional order and time-delay is described. The Caputo

sense definition applies to the variable fractional order derivative. The generalized polynomials with unknown coefficients are used to deconstruct the approximate solution of the suggested problem. The variable fractional order time-delay diffusion equation will be turned into a system of algebraic equations using the Shifted Chebyshev-Laguerre Operational Matrices. By employing Newton's iterative method, we get the values of the unknown coefficients. Some illustrative examples are given in order to prove the simplicity and accuracy of the proposed method.

Abbreviations

FC	fractional calculus
FPDEs	fractional partial differential equations
VOFPDEs	variable order fractional partial differential equations
HPM	Homotopy perturbation method
VOFc	variable order fractional calculus
VO	variable order
SLPs	shifted Legendre Polynomials
SℓPs	shifted Laguerre Polynomials
TPs	Taylor polynomials
SCPs	shifted Chebyshev polynomials
VFDDEs	variable order fractional delay diffusion equation
VFD	variable order fractional derivative
OMs	operational matrices
PDEs	partial differential equations
VOFDPDEs	variable order fractional delay partial differential equations
FTDDE	fractional time-delay diffusion equation
AE	Absolute error

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Chapter 1

Introduction

A more contemporary area of mathematical analysis and fractional calculus involves the derivatives and integrals of real or complex numbers and is an expanded version of integer differential calculus order [1, 2]. A more contemporary area of mathematical analysis, fractional calculus involves the derivatives and integrals of real or complex numbers and is an expanded version of integer differential calculus [3–9]. This is primarily caused by the truth that fractional operators assess the system's advancing by taking into account global correlation rather than just local characteristics. Additionally, derivatives of fractional order calculus may be more appropriate because integer order calculus occasionally contradicts the results of experiments [10–12]. In recent decades, fractional calculus FC has played a significant role in science and engineering, and therefore the scientists focused on its applications to model the real phenomena [1, 13–15]. The fractional derivative and integrals, were recognized to be an efficient tool to describe the properties of complex dynamical processes more accurately than the standard integer derivative and integral [12, 16–19]. Recently, more and more Numerous significant dynamical problems are showing fractional order behavior that may change across time or space, according to researchers. Variable order calculus is a new idea that has only recently come into existence parading in science [20]. Samko [14, 21], exhibited some features and an inversion formula while immediately generalizing the Riemann-Liouvill and Marchaud fractional integration and differentiation of the case of variable order. Moreover, Lorenzo and Hartled [22] and Coimbra [23], put the mathematics of variable order fractional calculus VOFc in perspective by discussing possible applications of VOFc in mechanics. Those works marked the starting point for application of variable order VO operators to the analysis of different complex physical problems. Studying partial differential equations of fractional order FPDEs motivated us with many applications. Fractional partial differential equations FPDEs are a fascinating subject because they are frequently

used to explain a variety of phenomena in real-world situations, including signal processing control theory, fluid flow, potential theory, information theory, finance, and entropy [24–27]. The memory term in FPDEs totally distinguishes them from PDEs of integer order, and solving FPDEs numerically or analytically is more difficult than solving PDEs. However, there are benefits to using the memory term in integral form, and it can be helpful for simulating physical or chemical phenomena when the most recent data is totally dependent on the most recent data from the entire past time [28]. Fractional partial differential equations of variable order VOFPDEs can be considered as a generalization to the FPDEs and due to the complexity of VOFPDEs are hard to obtain. Most researchers used different methods to solve (VOFPDEs) such as Bernstein polynomials [29], Finite difference method [30] and accurate spectral method [31], etc.
It is commonly recognized that delay plays a crucial role in modeling some processes and dynamical systems in the actual world [32, 33]. Yet, there are still only a few literary works that are devoted to the VOFPDEs with time delay [34–36]. To fill this gap we will solve in this thesis variable order fractional time-delay diffusion equations using collocation method. Many models of specific processes or dynamical systems in real-world problems exhibit neutral delay, which is always described using delay differential equations DDEs or time delay systems [33, 37, 38]. Despite the fact that FPDEs have been considered by a few researchers [39–41] and the references therein, there has been no work done in the area of VFDPDEs to our knowledge. One of the most important methods used in solving the equation is the subject of study in this thesis Legendre polynomials, Laguerre polynomials and Chebyshev polynomials [42–46]. Therefore, this reason motivates us in this thesis to propose a numerical technique to solve a class of VFDDEs using the collocation method and the OMs of the shifted Legendre-Laguerre polynomials.
A considerable advantage of the method is that the shifted Legendre-Laguerre polynomial coefficients of the solution are found very easily by using computer programs. Also according to the proposed model the time of the occurrence of an event dose not have fix domain. So for approximate the time functions in the problem, we applying the Laguerre polynomials, which defined in $[0, \infty)$.
Finally by using a few terms of shifted Legendre-Laguerre functions approximate solution converges to the exact solution.

It is well known that in real world problems delay is important to model certain processes and dynamical systems [33, 37]. However, there are still few works in the literature dedicated to VFPDEs [26, 47, 48] up to the our knowledge there has been no works on VFPDEs with proportional delays. Therefore this encouraged us in this study to handle and finding the approximate solutions of the

VFPDEs with proportional delays using HPM with the aid of approximating the variable order fractional derivative using the approach given in [49]. HPM is a semi-analytic method for finding the approximate solutions of non-liner problems. It was suggested by He [50, 51], the Homotopy idea in topology is combined with conventional perturbation approach to HPM. Without linearization or discretization , HPM can provide both approximate and exact solutions. The application of the HPM has appeared in many papers actually during the recent years, which shows that the method is powerful technique for studying the approximate solutions [11, 15, 18, 50–56].

This thesis contains seven chapters. The introduction is the first chapter, and in the second chapter, the most important laws and definitions of fractional calculus and integration of fixed and variable order were presented, as well as some types of diffusion equations with variable order. Chapter 3: Solving a fractional time-delay diffusion equation with a variable-order derivative based on shifted Legendre-Laguerre operational matrices. Homotopy perturbation method for solving time-fractional nonlinear variable-order delay partial differential equations It was the fourth chapter. Chapter fifth Shifted Chebyshev operational matrices to solve the fractional time-delay diffusion equation In the sixth chapter, an amendment was made to the method used in the third chapter by replacing shifted Legendre operational matrices with shifted Chebyshev operational matrices. The seventh chapter contains conclusions and future work. The practical results in this dissertation were obtained using the MATLAB 2020 program.

Chapter 2

Preliminaries and Fundamental Concepts

2.1 Beta and Gamma functions

Gamma function is the extension of factorial function to real numbers, and it was first introduced by the Swiss mathematician Leonhard Euler in the 18^{th} century. Gamma function, is closely related to beta function, also known as Euler's integral of the first kind. Beta and Gamma both calculus relies heavily on functions since they allow for the moderation of complex integrals into simpler forms utilizing the Beta and Gamma functions [57].

2.1.1 Gamma function

The gamma function, denoted by $\Gamma(\xi)$, is defined by

$$\Gamma(\xi) = \int_0^\infty e^{-s} \, s^{\xi-1} \, ds, \tag{2.1}$$

the integral is converges only for $\xi > 0$, and a recurrence formula for Gamma function is

$$\Gamma(\xi+1) = \xi \, \Gamma(\xi), \tag{2.2}$$

where $\Gamma(1) = 1$. From (2.2), $\Gamma(\xi)$ can be determined for all $\xi > 0$, when the values for $1 \leq \xi < 2$ are known and if ξ is positive integer, then

$$\Gamma(\xi+1) = \xi!, \quad \xi = 1, 2, 3, ... \tag{2.3}$$

For this reason $\Gamma(\xi)$ is called the factorial function.

2.1.2 Beta function

The Beta function, denoted by $B(\xi, \tau)$ is defined by

$$B(\xi, \tau) = \int_0^1 s^{\xi-1}(1-s)^{\tau-1} ds, \tag{2.4}$$

which is convergent for $\xi > 0, \ \tau > 0$.
The Beta function is connected with the Gamma function according to the relation

$$B(\xi, \tau) = \frac{\Gamma(\xi)\Gamma(\tau)}{\Gamma(\xi+\tau)} \tag{2.5}$$

If we find $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{s}} e^{-s} ds = \sqrt{\pi}$, we can define $B(\xi, \tau)$ for $\xi < 0, \ \tau < 0$.
Many integrals can be evaluated in terms of beta or Gamma functions.

2.2 Fractional derivatives and Integrals

A fractional differential equation is an equation which, contains fractional derivatives, a fractional integral equation is an integral equation containing fractional integrals. A fractional-order system means a system described by a fractional differential equation, a fractional integral equation or by a system of each equations. Let us consider the infinite sequence of n-fold integrals and derivatives as

$$f(\tau), \ \int_a^\tau f(\tau)\, d\tau_1, \ \int_a^{\tau_2} d\tau_1,$$

$$f(\tau), \ \frac{df(\tau)}{d\tau}, \ \frac{d^2 f(\tau)}{d\tau^2},$$

The derivative of arbitrary real order δ can be considered as an interpolation of this sequence of operators, we will use for it the notation suggested and used by Davis [38], namely $D_\tau^\delta f(\tau)$. The short name for derivatives of arbitrary order is fractional derivatives. The subscripts δ and τ denoted the two limits related to the operation of fractional differential, we will call them terminals of fractional differential. The fractional integrals means integrals of arbitrary order.

2.3 Fractional Calculus

In this section, we introduce some basic concepts and definitions[1, 14]:
Let $L_1 = L_1[a, b]$ be the class of Lebesgue function on the interval [a,b], $0 \leq a < b < \infty$, with norm defined by

$$\|\psi\| = \int_a^b \psi(\tau)\, d\tau, \ \ \psi \in L_1 \tag{2.6}$$

Definition 2.1.

A real function $\psi(\tau),\ \tau > 0$, is said to be in the space $C_\mu,\ \mu \in R$, if there exist a number $p > \mu$, such that $\psi(\tau) = \tau^p \psi_1(\tau)$, where $\psi_1(\tau) \in C[0, \infty)$ [10].

Definition 2.2.

A real function $\psi(\tau),\ \tau > 0$, is said to be in the space $C_\mu^k \in N$, if $\psi^k \in C_\mu$ and let δ, β be two positive real number, then we have the following definitions for the functional (arbitrary) order integration [10].

Definition 2.3.

I^δ denotes the fractional integral operator of order δ in the sense of Riemann-Liouville, [10] is defined by

$$I^\delta \psi(\tau) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^\tau \frac{\psi(s)}{(\tau - s)^{1-\delta}}\, ds, & \delta > 0 \\ \\ \psi(\tau), & \delta = 0, \end{cases} \tag{2.7}$$

Definition 2.4.

${}_R D^\delta$ denotes the fractional differential operator of order δ in the sense of Riemann-Liouville, [1] is defined by

$${}_R D^\delta \psi(\tau) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \frac{d^m}{d\tau^m} \int_0^\tau \frac{\psi(s)}{(\tau - s)^{\delta - m + 1}}\, ds, & 0 \le m - 1 < \delta \le m \\ \\ \frac{d^m \psi(\tau)}{d\tau^m}, & \delta = m \in N, \end{cases} \tag{2.8}$$

or

$${}_R D^\delta \psi(\tau) = \frac{d^m}{d\tau^m} I^{m-\delta} \psi(\tau),\ m = 1, 2, \ldots. \tag{2.9}$$

Definition 2.5.

Let $\psi \in C_{-1}^m,\ m \in N$. Then the Caputo fractional derivative of $\psi(\tau)$, [1] defined by

$${}^C D^\delta \psi(\tau) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^\tau \frac{\psi^{(m)}(\tau)}{(\tau - s)^{\delta - m + 1}}\, ds, & 0 \le m - 1 < \delta \le m, \\ \\ \frac{d^m \psi(\tau)}{d\tau^m}, & \delta = m \in N, \end{cases} \tag{2.10}$$

or

$${}^C D^\delta \psi(\tau) = I^{m-\delta} \frac{d^m}{d\tau^m} \psi(\tau),\ \ m = 1, 2, \ldots\ . \tag{2.11}$$

Now, we introduce some basic properties of the fractional operators which are listed below [1]:

For $\psi \in C_\mu,\ \mu \geq -1,\ \gamma \geq -1,\ \delta,\ \beta \geq 0$;

$$
\begin{aligned}
&1.\ I^\delta I^\beta \psi(\tau) = I^{\delta+\beta}\psi(\tau) = I^\beta I^\delta \psi(\tau)\\
&2.\ I^\delta \tau^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\delta+\gamma+1)}\tau^{\delta+\gamma}\\
&3.\ D^\delta(I^\delta\psi(\tau)) = \psi(\tau)\\
&4.\ I^\delta(D^\delta\psi(\tau)) = \psi(\tau) - \sum_{k=0}^{m-1}\psi^{(k)}(0^+)\frac{\tau^k}{k!}, 0 \leq m-1 < \delta \leq m \in N\\
&5.\ (I^\beta)^n\psi(\tau) = I^{n\beta}\psi(\tau), \quad n = 1, 2, \ldots\ .
\end{aligned}
\tag{2.12}
$$

Lemma 2.1.

For γ, $\beta > 0$ and $\psi(\tau) \in L_1 = L_1[a, b]$, [1] then

$$I^\beta I^\gamma \psi(\tau) = I^\gamma I^\beta \psi(\tau) = I^{\gamma+\beta}\psi(\tau), \tag{2.13}$$

and

$$(I^\beta)^n \psi(\tau) = I^{n\beta}\psi(\tau),\ n = 1, 2, \ldots\ . \tag{2.14}$$

Remark 2.1.

There exists a relation between the Riemann-Liouville and Caputo fractional derivatives; $\delta \in (0, 1)$ yields[1]

$${}_*D_a^\delta \psi(\tau) = \frac{(\tau-a)^{-\delta}}{\Gamma(1-\delta)}\psi(a) + {}^cD_a^\delta\psi(\tau), \tag{2.15}$$

and when $\psi(a) = 0$, then

$${}_*D_a^\delta \psi(\tau) = {}^cD_a^\delta\psi(\tau).$$

Theorem 2.1.

Let δ, $\beta \in (0, 1]$ and $\beta \geq a$. If $\psi(\tau)$ is bounded, measurable and $I^\beta\psi(\tau)$ is absolutely continuous, [2] then

$${}^cD^\delta I^\beta \psi(\tau) = I^{\beta-\delta}\psi(\tau). \tag{2.16}$$

Theorem 2.2.

Let δ, $\beta \in (0, 1]$ and $\beta \geq a$. If $\psi(\tau) \in L_1$, [2] then

$${}_*D^\delta I^\beta \psi(\tau) = I^{\beta-\delta}\psi(\tau). \tag{2.17}$$

Theorem 2.3.

Let δ, $\beta \in (0,1]$. Let $\psi(\tau)$ be absolutely continuous function on [a,b].Then [2]

$$\begin{aligned}
&(i)\ \beta \leq \delta \Rightarrow I^{\beta}\ {}^{c}D^{\delta}\psi(\tau) = D^{\delta-\beta}\psi(\tau),\\
&(ii)\ If\ \psi(a) = 0 \quad then\\
&(a)\ {}^{c}D^{\delta}I^{\beta}\psi(\tau) = {}^{c}D^{\delta-\beta}\psi(\tau),\ \delta \geq \beta,\\
&(b)\ I^{\beta}\ {}^{c}D^{\delta}\psi(\tau) = I^{\beta-\delta}\psi(\tau),\ \beta \geq \delta.
\end{aligned} \tag{2.18}$$

Theorem 2.4.

Let δ, $\beta \in (0,1]$. Let $\psi(\tau)$ be absolutely continuous function on [a,b]. Then if one of the assumptions [2]
(i) $\delta + \beta \in (0,1]$ and $\psi'(\tau)$ is bounded,
(ii) $\delta + \beta \in (1,2]$, $\psi''(\tau)$ exists and $\psi'(a) = 0$
holds, we have

$$ {}^{c}D^{\delta}\,{}^{c}D^{\beta}\psi(\tau) = D^{\delta+\beta}\psi(\tau) \tag{2.19}$$

Theorem 2.5.

Let $\delta \in (0,1]$. $f(\tau)$ is an absolutely continuous function on [a,b], then [2]

$$\begin{aligned}
&(i)\ I_a^{\delta}\,{}^{c}D_a^{\delta}\psi(\tau) = \psi(\tau) - \psi(a),\\
&(ii)\ {}^{c}D_a^{\delta}I_a^{\delta}\psi(\tau) = \psi(\tau).
\end{aligned} \tag{2.20}$$

2.4 Variable Fractional Order

Our goal in this section is to consider fractional derivatives of variable-order, with δ depending on time τ and variable ξ. In fact, some phenomena in physics are better described when the order of the fractional operator is not constant, for example, in the diffusion process in an inhomogeneous or heterogeneous medium, or processes where the changes in the environment modify the dynamic of the particle. Motivated by the above considerations, we introduce three types of Caputo fractional derivatives. The order of the derivative is considered as a function $\delta(\tau)$ taking values on the open interval $(0,1)$. To start, we define two different kinds of Riemann–Liouville fractional derivatives [21, 58].

Definition 2.6.

Given a function $\psi : [a,b] \to R$:

1. The type I left Riemann-Liouville fractional derivative of order $\delta(\tau)$ is defined by

$$ {}_aD_\tau^{\delta(\tau)}\ \psi(\tau) = \frac{1}{\Gamma(1-\delta(\tau))}\frac{d}{d\tau}\int_a^\tau (\tau - s)^{-\delta(\tau)}\ \psi(s)\,ds. \tag{2.21} $$

2. The type I right Riemann–Liouville fractional derivative of order $\delta(\tau)$ is defined by

$$ {}_bD_\tau^{\delta(\tau)}\ \psi(\tau) = \frac{-1}{\Gamma(1-\delta(\tau))}\frac{d}{d\tau}\int_\tau^b (s - \tau)^{-\delta(\tau)}\ \psi(s)\,ds. \tag{2.22} $$

3. The type II left Riemann–Liouville fractional derivative of order $\delta(\tau)$ is defined by

$$ {}_aD_\tau^{\alpha(\tau)}\ \psi(\tau) = \frac{d}{d\tau}(\frac{1}{\Gamma(1-\delta(\tau))}\int_a^\tau (\tau - s)^{-\alpha(\tau)}\ \psi(s)\,ds). \tag{2.23} $$

4. The type II right Riemann–Liouville fractional derivative of order $\delta(\tau)$ is defined by

$$ {}_bD_\tau^{\delta(\tau)}\ \psi(\tau) = \frac{d}{d\tau}(\frac{-1}{\Gamma(1-\delta(\tau))}\int_\tau^b (s - \tau)^{-\delta(\tau)}\ \psi(s)\,ds), \tag{2.24} $$

the Caputo derivatives are given using the previous Riemann–Liouville fractional derivatives.

Definition 2.7.

Given a function $\psi : [a,b] \to R$:

1. The type I left Caputo derivative of order $\delta(\tau)$ is defined by

$$ \begin{aligned} {}_a^cD_\tau^{\delta(\tau)}\ \psi(\tau) &= {}_aD_\tau^{\delta(\tau)}(\psi(\tau) - \psi(a)) \\ &= \frac{1}{\Gamma(1-\delta(\tau))}\frac{d}{d\tau}\int_a^\tau (\tau - s)^{-\delta(\tau)}[\psi(s) - \psi(a)]\,ds. \end{aligned} \tag{2.25} $$

2. The type I right Caputo derivative of order $\delta(\tau)$ is defined by

$$ \begin{aligned} {}_\tau^cD_b^{\delta(\tau)}\ \psi(\tau) &= {}_\tau D_b^{\delta(\tau)}(\psi(\tau) - \psi(b)) \\ &= \frac{-1}{\Gamma(1-\delta(\tau))}\frac{d}{d\tau}\int_\tau^b (s - \tau)^{-\delta(\tau)}[\psi(s) - \psi(b)]\,ds. \end{aligned} \tag{2.26} $$

3. The type II left Caputo derivative of order $\delta(\tau)$ is defined by

$$
\begin{aligned}
{}_{a}^{c}D_{\tau}^{\delta(\tau)}\ \psi(\tau) &= {}_{a}D_{\tau}^{\delta(\tau)}(\psi(\tau)-\psi(a)) \\
&= \frac{d}{d\tau}\Big(\frac{1}{\Gamma(1-\delta(\tau))}\int_{a}^{\tau}(\tau-s)^{-\delta(\tau)}[\psi(s)-\psi(a)]\,ds\Big).
\end{aligned}
\tag{2.27}
$$

4. The type II right Caputo derivative of order $\delta(\tau)$ is defined by

$$
\begin{aligned}
{}_{t}^{c}D_{b}^{\delta(\tau)}\ \psi(\tau) &= {}_{\tau}D_{b}^{\delta(\tau)}(\psi(\tau)-\psi(b)) \\
&= \frac{d}{d\tau}\Big(\frac{-1}{\Gamma(1-\delta(\tau))}\int_{\tau}^{b}(s-\tau)^{-\delta(\tau)}[\psi(s)-\psi(b)]\,ds\Big).
\end{aligned}
\tag{2.28}
$$

5. The type III left Caputo derivative of order $\delta(\tau)$ is defined by

$$
{}_{a}^{c}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \frac{1}{\Gamma(1-\delta(\tau))}\int_{a}^{\tau}(\tau-s)^{-\delta(\tau)}\acute{\psi}(s)\,ds. \tag{2.29}
$$

6. The type III right Caputo derivative of order α is defined by

$$
{}_{\tau}^{c}D_{b}^{\delta(\tau)}\ \psi(\tau) = \frac{-1}{\Gamma(1-\delta(\tau))}\int_{\tau}^{b}(s-\tau)^{-\delta(\tau)}\acute{\psi}(s)\,ds. \tag{2.30}
$$

Definition 2.8.

The left and right Riemann–Liouville fractional derivatives of order $\delta(\cdot,\cdot)$ are defined by

$$
{}_{a}D_{\tau}^{\delta(\cdot,\cdot)}\ \psi(\tau) = \frac{d}{d\tau}\int_{a}^{\tau}\frac{1}{\Gamma(1-\delta(\tau,s))}\ (\tau-s)^{-\delta(\tau,s)}\ \psi(s)\,ds\ , \quad \tau > a \tag{2.31}
$$

and

$$
{}_{b}D_{\tau}^{\delta(\cdot,\cdot)}\ \psi(\tau) = \frac{d}{d\tau}\int_{\tau}^{b}\frac{-1}{\Gamma(1-\delta(s,\tau))}\ (s-\tau)^{-\delta(s,\tau)}\ \psi(s)\,ds\ , \quad \tau < b, \tag{2.32}
$$

respectively.

Definition 2.9.

The left and right Caputo fractional derivatives of order $\delta(\cdot,\cdot)$ are defined by [59]

$$
{}_{a}^{c}D_{\tau}^{\delta(\cdot,\cdot)}\ \psi(\tau) = \int_{a}^{\tau}\frac{1}{\Gamma(1-\delta(\tau,s))}\ (\tau-s)^{-\delta(\tau,s)}\ \psi^{(1)}(s)\,ds\ , \quad \tau > a \tag{2.33}
$$

and

$$
{}_{b}^{c}D_{\tau}^{\delta(\cdot,\cdot)}\ \psi(\tau) = \int_{\tau}^{b}\frac{1}{\Gamma(1-\delta(s,\tau))}\ (s-\tau)^{-\delta(s,\tau)}\ \psi^{(1)}(s)\,ds\ , \quad \tau < b, \tag{2.34}
$$

respectively.

Definition 2.10.

Let $\psi \in H^1(a,b), a < \tau < b$ and $M(\delta)$ be a normalization function such that $M(0) = M(1) = 1$, then

1. The left Caputo-Fabrizio derivative of variable-order fractional $\delta(\tau)$ is defined by

$$a^{CF}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \frac{M(\delta(\tau))}{1-\delta(\tau)} \int_a^{\tau} exp(-\delta(\tau)\frac{\tau - s}{1-\delta(\tau)})\ \psi^{(1)}(s)\, ds, \quad \tau > a. \tag{2.35}$$

2. The left Caputo-Fabrizio derivative of variable-order fractional $\delta(\tau)$ is defined by

$${}_{b}^{CF}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \frac{M(\delta(\tau))}{1-\delta(\tau)} \int_{\tau}^{b} exp(-\delta(\tau)\frac{s - \tau}{1-\delta(\tau)})\ \psi^{(1)}(s)\, ds, \qquad \tau < b. \tag{2.36}$$

Definition 2.11.

The Atangana-Baleanu fractional derivatives in the caputo sense(ABC derivative) of variable order $\delta(\tau)$ for given function $\psi(\tau) \in H^1(a,b), b > \tau > a$ (where C denotes Caputo) with base point a is defined at a point $\tau \in (a,b)$.

1. The type I partial left Atangana-Baleanu-Caputo derivative:

$${}_{a}^{ABC}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \frac{B(\delta(\tau))}{1-\delta(\tau)} \int_a^{\tau} \acute{\psi}(s)\ E_{\delta(\tau)}[-\delta(\tau)\frac{(\tau - s)^{\delta(\tau)}}{1-\delta(\tau)}]\, ds. \tag{2.37}$$

They suggested that B has the same properties as in Caputo and Fabrizio case.

2.The type I partial right Atangana-Baleanu-Caputo derivative:

$${}_{b}^{ABC}D_{\tau}^{\delta(t)}\ \psi(\tau) = \frac{B(\delta(\tau))}{1-\delta(\tau)} \int_{\tau}^{b} \acute{\psi}(s)\ E_{\delta(\tau)}[-\delta(\tau)\frac{(s - \tau)^{\delta(\tau)}}{1-\delta(\tau)}]\, ds. \tag{2.38}$$

3. The type II partial left Atangana-Baleanu-Caputo derivative:

$${}_{a}^{ABC}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \int_a^{\tau} \frac{B(\delta(s))}{1-\delta(s)}\ \acute{\psi}(s)\ E_{\delta(s)}[-\delta(s)\frac{(\tau - s)^{\delta(s)}}{1-\delta(s)}]\, ds. \tag{2.39}$$

4. The type II partial right Atangana-Baleanu-Caputo derivative:

$${}_{b}^{ABC}D_{\tau}^{\delta(\tau)}\ \psi(\tau) = \int_{\tau}^{b} \frac{B(\delta(s))}{1-\delta(s)}\ \acute{\psi}(s)\ E_{\delta(s)}[-\delta(s)\frac{(s - \tau)^{\delta(s)}}{1-\delta(s)}]\, ds. \tag{2.40}$$

2.5 Delay partial differential Equations

In economics, physics, chemistry, biology, medicine, and engineering, there are numerous natural and manmade physical processes that entail time delays. Differential equations with time delays are frequently referred to as time-delay systems or delay differential equations DDEs. DDEs have been used to represent phenomena in many different scientific domains, including economics, physics, ecology, engineering control, and nuclear engineering. Few DDEs whose analytical solutions may be described directly exist, as is widely known, such as the method of steps or the Laplace transform method. Due to a lack of theoretical research in this field, many numerical methods have been presented recently. Given the significance of fractional calculus, delay fractional differential equations DFDEs research should be given attention. Numerical and analytical elementsAlthough many people have thought about delay partial differential equations, We will discuss several partial differential equations ofvariable-order fractional with delay

1. Korteweg-de Vries (KdV) equation [60]

$$ {}_{a}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) + b\nu\ \nu_{\xi}(p_0\xi, q_0\tau) + \nu_{\xi\xi\xi}(p_1\xi, q_1\tau) = 0, \quad 0 < \delta(\xi,\tau) \leq 1, \tag{2.41} $$

where b is a constant, which come up when analyzing waves in shallow water.

2. Klein-Gordon equation [61]

$$ {}_{a}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \nu_{\xi\xi}(p_0\xi, q_0\tau) - b\nu(p_1\xi, q_1\tau) - F(\nu(p_2\xi, q_2\tau)) + h(\xi,\tau), \ 1 < \delta(\xi,\tau) \leq 2, \tag{2.42} $$

where b is a constant, $h(\xi,\tau)$ is a known analytical function and $F(\nu(\xi,\tau))$ is a nonlinear function of $\nu(\xi,\tau)$ it describes the interaction of nonlinear waves and stems from quantum field theory.

3. The One dimensional VO Diffusion equation [44]

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \frac{\partial^2\nu(\xi,\tau)}{\partial\xi} + F(\xi,\tau-k), \ \ (\xi,\tau)\in[0,h]\times[0,t], \ 0 < \delta(\xi,\tau) \leq 1, \tag{2.43} $$

where k is the value of delay and the initial condition

$$ \nu(\xi,0) = g_0(\xi), \quad 0 < \xi < h, \tag{2.44} $$

and boundary conditions

$$ \nu(0,\tau) = g_1(\tau), \quad \nu(h,\tau) = g_2(\tau), \quad 0 < \tau < t. \tag{2.45} $$

4. The two dimensional VO Diffusion equation [44]

$$
{}_0^C D_\tau^{\delta(\xi,\vartheta,\tau)}\nu(\xi,\vartheta,\tau) = \frac{\partial^2\nu(\xi,\vartheta,\tau)}{\partial\xi^2} + \frac{\partial^2\nu(\xi,\vartheta,\tau)}{\partial\vartheta^2} + F(\xi,\vartheta,\tau-k),
$$
$$
(\xi,\vartheta,\tau) \in [0,h_1]\times[0,h_2]\times[0,t], \quad 0<\delta(\xi,\vartheta,\tau)\le 1, \tag{2.46}
$$

with initial condition

$$
\nu(\xi,\vartheta,0) = g_0(\xi,\vartheta), \quad (\xi,\vartheta)\in[0,h_1]\times[0,h_2], \tag{2.47}
$$

and four boundary conditions

$$
\begin{aligned}
&\nu(0,\vartheta,\tau) = g_1(\vartheta,\tau), \quad \nu(h_1,\vartheta,\tau) = g_2(\vartheta,\tau), \quad (\vartheta,\tau)\in[0,h_2]\times[0,t],\\
&\nu(\xi,0,\tau) = q_1(\xi,\tau), \quad \nu(\xi,h_2,\tau) = q_2(\xi,\tau), \quad (\xi,\tau)\in[0,h_1]\times[0,t].
\end{aligned} \tag{2.48}
$$

5. Newell-Whitehead equation [62]

$$
{}_0^{ABC} D_\tau^{\delta(\xi,\tau)}\nu(\xi,\tau) = \nu_{\xi\xi}(\xi,\tau) + \nu(\xi,\tau) - \nu^3(\xi,\tau-k), \quad (\xi,\tau)\in[-10,10]\times[0,1], \tag{2.49}
$$

the closed-form solution in the case of $\delta(\xi,\tau)=1$, which can be determined according to determine the required initially and boundary conditions as:

$$
\nu(\xi,\tau) = 0.5(1+tanh(\frac{\sqrt{2}\xi+3\tau}{4}))
$$

6. Fitzhugh-Nagumo equation [62]

$$
{}_0^{ABC} D_\tau^{\delta(\xi,\tau)}\nu(\xi,\tau) = \nu_{\xi\xi}(\xi,\tau) + \nu(\xi,\tau)(1-\nu(\xi,\tau))(\nu(\xi,\tau-k)), \tag{2.50}
$$

the closed-form solution in the case of $\delta(\xi,\tau)=1$, which can determined according the required initially and boundary conditions as:

$$
\nu(\xi,\tau) = (1+e^{\frac{-1}{\sqrt{2}}(\xi+\frac{(1-2k)\tau}{\sqrt{2}})})^{-1}
$$

2.6 Orthogonal polynomials

2.6.1 The shifted Legendre polynomials

The recurrence relation below defines the Legendre polynomials on $[-1,1]$ [57].

$$
L_{i+1}(\zeta) = \frac{2i+1}{i+1}\,\zeta\,L_i(\zeta) - \frac{i}{i+1}L_{i-1}(\zeta), \quad i=1,2,...,
$$

where $L_0(\zeta) = 0, \quad L_1(\zeta) = \zeta.$

The transformation $\xi = \frac{\zeta+1}{2}$ changes interval $[-1, 1]$ to $[0, 1]$ as well as shifted Legendre polynomials are provided

$$\phi_i(\xi) = \sum_{\kappa=0}^{i} (-1)^{i+\kappa} \frac{(i+\kappa)!\ \xi^{\kappa}}{(i-\kappa)\ (\kappa!)^2}, \quad i = 0, 1, 2, ...,$$

where $\phi_i(\xi) = (-1)^i, \quad \phi_1(\xi) = 1.$
The orthogonality condition is

$$\int_0^1 \phi_i(\xi)\ \phi_j(\xi) d\xi = \begin{cases} \frac{1}{2i+1}, & if \quad i = j \\ 0, & if \quad i \neq j, \end{cases}$$

meaning that any function $\psi(\xi) \in C[0, 1]$, can be approximated by using shifted Legendre polynomials as:

$$\psi(\xi) \approx \sum_{b}^{n} C_b \phi_b(\xi), \quad where \quad C_b = (2b+1) \int_0^1 \psi(\xi) \phi_b(\xi) d\xi.$$

In vector notation, we write

$$\psi(\xi) \approx K_N \hat{\phi_N}(\xi)$$

where $N = n+1, K$ is the coefficient vector and $\hat{\phi_N}(\xi)$ is N terms function vector.

2.6.2 Shifted Chebyshev polynomials

The nth SCHP,$\mathsf{T}_n(\xi)$, using recursive expressions, the following result is obtained [63]:

$$\mathsf{T}_0(\xi) = 1, \quad \mathsf{T}_1(\xi) = 2\xi - 1,$$

$$\mathsf{T}_{n+1}(\xi) = 2(2\xi - 1)\ \mathsf{T}_n\ (\xi) - \mathsf{T}_{n-1}(\xi), \quad n \in N.$$

In fact, the set $\{\mathsf{T}_n(\xi)\}_{n=0}^{\infty}$ is constituted by orthogonal functions having for weight function $\omega(\xi) = \frac{1}{\sqrt{1-(2\xi-2)^2}}, \quad \xi \in [0, 1]$. Then for any arbitrary function $\psi(\xi) \in L^2_{\omega}[0, 1]$, it can be approximated by using Chebyshev polynomials as:

$$\psi(\xi) = \sum_{n=0}^{\infty} A_n\ \mathsf{T}_n\ (\xi)$$

where the coefficients A_n is calculable as

$$A_n = \frac{4}{\pi \gamma_n} \int_0^1 \psi(\xi)\ \mathsf{T}_n(\xi)\ \omega(\xi) d\xi,$$

for $\gamma_0 = 2, \quad \gamma_n = 1, \quad n \geq 1$,and the first $N+1$ terms gives

$$\psi(\xi) \simeq \sum_{n=0}^{N} A_n\ \mathsf{T}_n(\xi) = C^T \varphi(\xi)$$

where $C = [A_0,\ A_1, ..., A_N]^T$ and $\varphi(\xi) = [\mathsf{T}_0(\xi),\ \mathsf{T}_1(\xi), ..., \mathsf{T}_N(\xi)]^T$.

2.6.3 Laguerre polynomials

The Laguerre polynomials is denoted by, $\ell_m(\xi), \quad (m \geq 1)$ and the generating function defines

$$\ell_m(\xi) = \sum_{i=0}^{m} \frac{(-1)^i \begin{pmatrix} m \\ i \end{pmatrix} \xi^i}{i!}, \quad m \geq 0$$

As an alternative, the recurrence relation also defines the Laguerre polynomials as follows:

$$\ell_0(\xi) = 1, \quad \ell_1(\xi) = 1-\xi, \quad (m+1)\ell_{m+1}(\xi) = (2m+1-\xi)\ell_m(\xi) - m\ell_{m-1}(\xi), \quad (m \geq 1).$$

The Laguerre polynomials Rodrigue's formula is provided by

$$\ell_m(\xi) = \frac{1}{m!}\ e^{\xi}\ \frac{d^m}{d\xi^m}(e^{-\xi}\ \xi^m), \qquad (m \geq 0).$$

Then for any arbitrary function $\psi(\xi) \in L^2_\omega[0,1]$ it can be approximated by using Laguerre polynomials as

$$\psi(\xi) = \sum_{m=0}^{\infty} C_m \ell_m(\xi),$$

where C_m are the unknown Laguerre coefficients is given by

$$C_m = \frac{\Gamma(m)}{\Gamma(m+1)} \int_0^{\infty} e^{-\xi} \ell_m(\xi) \psi(\xi) d\xi, \quad m = 0, 1, ...$$

Chapter 3

Solving Fractional Time-Delay Diffusion Equation with Variable-Order Derivative Based on Shifted Legendre-Laguerre Operational Matrices

This chapter consists of eight sections. In section one a literature survey about the fractional time-delay diffusion equation is given, while section two related to the problem statement. In section three the function approximation is given. Sections four and five are about the operational matrices of integral (integer and variable) of the shifted Legendre and Lagurre polynomials, our approach will be presented in section six, convergence analysis is given in section seven, finally numerical examples are considered in section eight.

3.1 Fractional Time-Delay diffusion equation

The Fractional Time-Delay Diffusion Equation FTDDE is a partial differential equation that simulates anomalous diffusion phenomena, such as non-linear or non-Gaussian diffusion processes that take place in the presence of external fields or inhomogeneities. The temporal derivative is changed to a fractional derivative of any order, generally represented by the Caputo derivative, in this modification of the basic diffusion equation. The memory effect of the diffusing particle is described by a delay term in the FTDDE. It stands for the lag in time between the system's apparent change and the cause of it. The delay term comprises the impacts of the external forces acting on the system as well as the observable time history of the system.

Numerous research and engineering sectors, including chemical engineering, finance, and biology, where anomalous diffusion phenomena are seen, can benefit from the FTDDE. Numerical techniques such the finite difference method, the finite element approach, or the spectral method are used to arrive at the numerical solution to the FTDDE [44, 62, 64, 65]. However, these techniques can be computationally expensive, particularly when dealing with complex geometries or higher orders of fractional derivatives. The FTDDE has been shown to be a potent tool for the modeling and study of anomalous diffusion processes, making it a significant area of research in the field of fractional calculus. To comprehend these occurrences and discover new uses, several researchers are exploring the FT-DDE and its extensions. Numerous academics have investigated equations with variable fractional order hysteresis. Dumitru Baleanu is a renowned researcher in this area. The existence and uniqueness of solutions for a class of variable order hysteresis equations with Caputo fractional derivative were examined by Baleanu and his colleagues in a research published in 2019 [66, 67]. Karel Van Bockstal and Mahmoud A. Zaky they presented a recent study to solve this equation [59]. The study also looked at the solution's characteristics and how they connected to the appropriate operator semigroup.

3.2 Problem statement

In this section, we state the variable order fractional time-delay diffusion equation that will be handled and analyzed in the next sections as follows:

$$D_{\tau}^{\delta(\xi,\tau)}\ \nu(\xi,\tau) - \eta\ \frac{\partial^2 \nu(\xi,\tau)}{\partial\ \xi^2} = f(\tau, \nu(\xi,\tau), \nu(\xi,\tau-\kappa)), \tag{3.1}$$

$$0 \le \xi \le 1, \quad 0 \le \tau < \infty.$$

Subject to :

$$\nu(0,\tau) = \nu_0(\tau), \quad \nu(1,\tau) = \nu_1(\tau). \tag{3.2}$$

and

$$\nu(\xi,0) = g_0(\xi), \quad \frac{\partial \nu(\xi,0)}{\partial \tau} = g_1(\xi). \tag{3.3}$$

So that, $\nu(\xi,\tau)$ is an unknown function, the known functions $\nu_0(\tau), \nu_1(\tau)$, $g_0(\xi)$ *and* $g_1(\xi)$ are given continuous functions. Also, $q = max_{(\xi,\tau)\in\Omega}\{\delta(\xi,\tau)\}$ *and* $q \in Z^+$.

3.3 Function Approximation

Consider the basis function $\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)$ which is two variable function and can be expanded as:

$$\Phi_{\tilde{m}\tilde{n}}(\xi,\tau) = G_{\tilde{m}}(\xi)\ \ell_{\tilde{n}}(\tau), \quad (\xi,\tau) \in \Omega = [0,1] \times [0,\infty), \tag{3.4}$$

where $\tilde{m} = 0,1,...,\tilde{M}, \quad \tilde{n} = 0,1,...,\tilde{N}, G_{\tilde{m}}(\xi)$ is the shifted Legendre polynomials defined on the interval $[0,1]$ and $\ell_n(\tau)$ is the shifted Laguerre polynomials defined on the interval $[0,\infty)$.

The shifted Legendre-Lagurre functions are orthogonal with respect to the weight function $\Xi(\xi,\tau) = exp(-\tau)$ in the Ω with the orthogonal property [45, 68, 69]:

$$\int_0^\infty \int_0^1 \Xi(\xi,\tau)\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)\Phi_{ij}(\xi,\tau)\ d\xi d\tau = \frac{1}{2\tilde{m}+1}\delta_{\tilde{m}i}\delta_{\tilde{n}j}. \tag{3.5}$$

where $\delta_{\tilde{m}i}$ and $\delta_{\tilde{n}j}$ are the Kronecker functions.

Any function $\nu(\xi,\tau) \in L_2(\Omega)$ may be decomposed as:

$$\nu(\xi,\tau) = \sum_{\tilde{m}=0}^{\tilde{M}} \sum_{\tilde{n}=0}^{\tilde{N}} \nu_{\tilde{m}\tilde{n}} \Phi_{\tilde{m}\tilde{n}}(\xi,\tau) \simeq G^T(\xi)V\ \ell(\tau), \tag{3.6}$$

where

$$\nu_{\tilde{m}\tilde{n}} = (2\tilde{m}+1)\int_0^\infty \int_0^1 \Xi(\xi,\tau)\nu(\xi,\tau)\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)d\xi d\tau. \tag{3.7}$$

and

$$V = \begin{bmatrix} \nu_{00} & \nu_{01} & \nu_{02} & \cdots & \nu_{0\tilde{N}} \\ \nu_{10} & \nu_{11} & \nu_{12} & \cdots & \nu_{1\tilde{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{\tilde{M}0} & \nu_{\tilde{M}1} & \nu_{\tilde{M}2} & \cdots & \nu_{\tilde{M}\tilde{N}} \end{bmatrix},$$

$$G(\xi) = [G_0(\xi),\ G_1(\xi), \ldots G_{\tilde{M}}(\xi)]^T and\ \ \ell(\tau) = [\ell_0(\tau),\ \ell_1(\tau), \ldots \ell_{\tilde{N}}(\tau)]^T. \tag{3.8}$$

3.4 Pseudo-Operational Matrix of Integer Order Integral of the shifted Legendre and Laguerre Polynomials:

The purpose of this section is to find the OMs of the integer order quintessential of SLPs and the SℓPs respectively using Taylor polynomials TPs [70–72], which is described as follows

$$T_k(\xi) = \xi^k, \qquad k = 0, 1, ..., M.$$

The SLPs may be expressed by means of the TPs as:

$$G(\xi) = D_1 \; T(\xi),$$

since

$$T(\xi) = [1, \xi, \xi^2, \ldots, \xi^M]^T, \qquad D_1 = [d^1_{\imath\jmath}]_{(M+1)\times(M+1)},$$

$$d^1_{\imath\jmath} = \begin{cases} \dfrac{(-1)^{\imath+\jmath} \; (\imath+\jmath)!}{(\imath-\jmath)! \; (\jmath!)^2}, \imath \geq \jmath \\ 0, \qquad\qquad otherwise. \end{cases} \tag{3.9}$$

Then, by integrating $G(\xi)$, the pseudo-operational matrix of the SLPs is obtained:

$$\int_0^\xi G(\rho) \; d\rho = \int_0^\xi D_1 T(\rho) d\rho = D_1 \int_0^\xi T(\rho) d\rho$$
$$= \xi D_1 \Lambda_1 T(\xi) = \xi D_1 \Lambda_1 D_1^{-1} G(\xi) = \xi \vartheta_1 G(\xi).$$

where $\vartheta_1 = D_1 \Lambda_1 D_1^{-1}$ is the pseudo-operational matrix of the integer order integral of the SLPs andΛ_1 is defind by,[42, 65, 73]:

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1/2 & 0 & ... & 0 \\ 0 & 0 & 1/3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(M+1) \end{bmatrix},$$

Similarly

$$\ell(\tau) = D_2 T(\tau), \tag{3.10}$$

where

$$d^2_{\imath\jmath} = \begin{cases} \dfrac{(-1)^{\imath} \; (\imath)!}{(\imath-\jmath)! \; (\jmath!)^2}, \imath \geq \jmath \\ 0, \qquad\qquad otherwise. \end{cases} \tag{3.11}$$

Also, with the aid of integrating $\ell(\tau)$,we gain the operational matrix of integer integration of the SℓPs as:

$$\int_0^\tau \ell(\rho)\, d\rho = \int_0^\tau D_2 T(\rho) d\rho = D_2 \int_0^\tau T(\rho) d\rho$$
$$= \tau D_2 \Lambda_2 T(\tau) = \tau D_2 \Lambda_2 D_2^{-1} \ell(\tau) = \tau \vartheta_2 \ell(\tau).$$

where $\vartheta_2 = D_2 \Lambda_2 D_2^{-1}$ is the pseudo operational matrix of the integer order integral of the shifted (SℓPs) and Λ_2 is given by:

$$\Lambda_2 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1/2 & 0 & ... & 0 \\ 0 & 0 & 1/3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(N+1) \end{bmatrix},$$

3.5 Pseudo Operational matrix of the variable order fractional integral of the (SℓPs):

To obtain the operational matrix of the variable order Riemann-Liouville fractional integration of order $\delta(\xi,\tau) > 0$ of the vector $\ell(\tau)$ defined in equation (3.10), we need to calculate first the variable order Riemann-Liouville fractional integral of the TPs which is written as:

$$I_\tau^{\delta(\xi,\tau)}\, T(\tau) = \tau^{\delta(\xi,\tau)}\, \gamma_N^{\delta(\xi,\tau)} T(\tau), \tag{3.12}$$

where

$$\gamma_N^{\delta(\xi,\tau)} = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(1+\delta(\xi,\tau))} & 0 & 0 & ... & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2+\delta(\xi,\tau))} & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \frac{\Gamma(N+1)}{\Gamma(N+1+\delta(\xi,\tau))} \end{bmatrix}.$$

Also, we need to find

$$I_\tau^{\delta(\xi,\tau)}\, \tau T(\tau) = \tau^{1+\delta(\xi,\tau)}\, \hat{\gamma}_N^{\delta(\xi,\tau)} T(\tau), \tag{3.13}$$

where

$$\hat{\gamma}_N^{\delta(\xi,\tau)} = \begin{bmatrix} \frac{\Gamma(2)}{\Gamma(2+\delta(\xi,\tau))} & 0 & 0 & ... & 0 \\ 0 & \frac{\Gamma(3)}{\Gamma(3+\delta(\xi,\tau))} & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \frac{\Gamma(N+2)}{\Gamma(N+2+\delta(\xi,\tau))} \end{bmatrix}.$$

Lemma 3.1.

Let $\ell(\tau)$ be the SℓPs vector defined in (3.10) and $q-1 < \delta(\xi,\tau) \leq q \in Z^+$ the pseudo-operational matrix of variable-order fractional integration of the vector $\ell(\tau)$ can be expressed as

$$I_\tau^{\delta(\xi,\tau)}\ell(\tau) = \tau^{\delta(\xi,\tau)}\ \Theta_N^{\delta(\xi,\tau)}T(\tau), \tag{3.14}$$

where

$$\Theta_N^{\delta(\xi,\tau)} = D_2\ \gamma_N^{\delta(\xi,\tau)}\ D_2^{-1}.$$

Proof:
A direct application of the relations (3.10) and (3.12) gives as:

$$\begin{aligned} I_\tau^{\delta(\xi,\tau)}\ \ell(\tau) &= I_\tau^{\delta(\xi,\tau)}D_2\ T(\tau) = \tau^{\delta(\xi,\tau)}D_2\ \gamma_N^{\delta(\xi,\tau)}T(\tau) \\ &= \tau^{\delta(\xi,\tau)}\ D_2\ \gamma_N^{\delta(\xi,\tau)}\ D_2^{-1}\ \ell(\tau) \\ &= \tau^{\delta(\xi,\tau)}\ \Theta_N^{\delta(\xi,\tau)}\ \ell(\tau). \end{aligned}$$

3.6 The Approach

This section is devoted to finding the numerical solution of the following VF-PDDEs in (3.1)-(3.3). For this problem assume that the easiest order of spinoff with appreciate to ξ and τ is 2. Therefore, we obtain the following approximate functions as

$$\frac{\partial^4\nu(\xi,\tau)}{\partial\xi^2\partial\tau^2} \simeq G^T(\xi)U\ell(\tau), \tag{3.15}$$

where the unknown matrix U is defined as follows:

$$U = \begin{bmatrix} u_{00} & u_{01} & u_{02} & \dots & u_{0N} \\ u_{10} & u_{11} & u_{12} & \dots & u_{1N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{M0} & u_{M1} & u_{M2} & \dots & u_{MN} \end{bmatrix}.$$

By integrating of the above equation (3.15) with respect to τ and using the initial condition (3.3), we have:

$$\frac{\partial^3\nu(\xi,\tau)}{\partial\xi^2\partial\tau} \simeq \tau G^T(\xi)U\vartheta_2\ell(\tau) + \acute{g}_1(\xi). \tag{3.16}$$

Integrating (3.16) with respect to τ, yields:

$$\frac{\partial^2\nu(\xi,\tau)}{\partial\xi^2} \simeq \tau^2 G^T(\xi)U\vartheta_2\hat{\vartheta}_2\ell(\tau) + \tau\acute{g}_1(\xi) + \acute{g}_0(\xi), \tag{3.17}$$

where

$$\begin{aligned}\int_0^\tau \rho L(\rho)d\rho &= \int_0^\tau \rho D_2 T(\rho)d\rho = D_2\int_0^\tau \rho T(\rho)d\rho \\ &= \tau^2 D_2\hat{\Lambda}_2 T(\tau) = \tau^2 D_2 \hat{\Lambda}_2 D_2^{-1}\ell(\tau) = \tau^2\hat{\vartheta}_2\ell(\tau),\end{aligned} \tag{3.18}$$

and

$$\hat{\Lambda}_2 = \begin{bmatrix} 1/2 & 0 & 0 & ... & 0 \\ 0 & 1/3 & 0 & ... & 0 \\ 0 & 0 & 1/4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(N+2) \end{bmatrix}.$$

Now, by integrating (3.17) with respect to ξ, we get

$$\begin{aligned}\frac{\partial \nu(\xi,\tau)}{\partial \xi} &\simeq \xi\ \tau^2 G^T(\xi)\vartheta_1^T U\vartheta_2\hat{\vartheta}_2\ell(\tau) + \tau(\acute{g}_1(\xi) - \acute{g}_0(0)) + (\acute{g}_0(\xi) - \acute{g}_0(0)) \\ &+ \frac{\partial \nu(0,\tau)}{\partial \xi},\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\nu(\xi,\tau) \simeq \xi^2\ \tau^2 G^T(\xi)\hat{\vartheta}_1^T\ \vartheta_1^T U\ \vartheta_2\hat{\vartheta}_2\ell(\tau) + \tau(g_1(\xi) - g_1(0) - \xi\ \acute{g}_1(0)) \\ + (g_0(\xi) - g_0(0) - \xi\ \acute{g}_0(0)) + \xi\ \frac{\partial \nu(\xi,0)}{\partial \xi} + \nu_0(\tau),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\int_0^\xi \rho P(\rho)d\rho &= \int_0^\xi \rho D_1 T(\rho)d\rho = D_1\int_0^\xi \rho T(\rho)d\rho \\ &= \xi^2 D_1\hat{\Lambda}_1 T(\xi) = \xi^2 D_1\hat{\Lambda}_1 D_1^{-1}G(\xi) = \xi^2\hat{\vartheta}_1 G(\xi),\end{aligned} \tag{3.21}$$

and

$$\hat{\Lambda}_1 = \begin{bmatrix} 1/2 & 0 & 0 & ... & 0 \\ 0 & 1/3 & 0 & ... & 0 \\ 0 & 0 & 1/4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(M+2) \end{bmatrix}.$$

Integrating (3.15) w.r.t. ξ and by the aid of the conditions (3.2) and (3.3), yields:

$$\frac{\partial^3 \nu(\xi,\tau)}{\partial \xi \partial \tau^2} \simeq \xi G^T(\tau)\vartheta_1^T U\ell(\tau) + \frac{\partial^3 \nu(0,\tau)}{\partial x\partial \tau^2}. \tag{3.22}$$

$$\frac{\partial^2 \nu(\xi,\tau)}{\partial \tau^2} \simeq \xi^2 G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ell(\tau) + \xi\ \frac{\partial^3 \nu(0,\tau)}{\partial \xi\partial \tau^2} + \acute{\nu_0}(\tau). \tag{3.23}$$

It is remarkable that $\frac{\partial^3\nu(0,\tau)}{\partial\xi\partial\tau^2}$ is unknown function, by integrating (3.22) from 0 to 1 with respect to ξ, we get:

$$\frac{\partial^3\nu(0,\tau)}{\partial\xi\partial\tau^2} \simeq \acute{\acute{\nu_1}}(\tau) - \acute{\acute{\nu_0}}(\tau) - S^T D_1^T \vartheta_1^T U\ell(\tau),$$

where

$$\int_0^1 \xi G^T(\rho)d\xi = \int_0^1 \xi T(\xi)D_1^T d\xi = S^T D_1^T,$$

and

$$S = [\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{M+2}]^T.$$

Then

$$\frac{\partial^2\nu(\xi,\tau)}{\partial\tau^2} \simeq \xi^2 G^T(\xi)\hat{\vartheta}_1^T \vartheta_1^T U\ell(\tau) + x[\acute{\acute{\nu_1}}(\tau) - \acute{\acute{\nu_0}}(\tau) - S^T D_1^T \vartheta_1^T U\ell(\tau)] + \acute{\acute{\nu_0}}(\tau). \tag{3.24}$$

By integrating (3.24)for τ, we acquire to

$$\frac{\partial\nu(\xi,\tau)}{\partial\tau} \simeq \xi^2 \tau G^T(\xi)\hat{\vartheta}_1^T \vartheta_1^T U\vartheta_2\ell(\tau) + \xi[\acute{\nu}_1(\tau) - \acute{\nu}_0(\tau) - \tau S^T D_1^T \vartheta_1^T U\vartheta_2\ell(\tau)] + \acute{\nu}_0(\tau) + g_1(\tau) \tag{3.25}$$

3.6.1 The Operational Matrix of the delay term:

In this subsection, the delay term $\nu(\xi, \tau-\kappa)$ will be approximated by using the operational matrix of the Laguerre polynomials as follows:
consider [42]:

$$\nu(\xi, \tau-\kappa) = G^T(\xi)U\ell(\tau-\kappa), \tag{3.26}$$

where

$$\ell(\tau-\kappa) = HP^T(\tau-\kappa), \tag{3.27}$$

and

$$H = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & ... & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & ... & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & ... & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}.$$

To get $P(\tau - k)$ by means of $P(\tau)$, we must employ the next relation:

$$P(\tau) = [1, \tau, \tau^2, ..., \tau^N], \quad P(\tau - \kappa) = [1, \tau - \kappa, (\tau - k)^2, ...(\tau - k)^N].$$

$$P(\tau - k) = P(\tau) B^T_{-k}, \tag{3.28}$$

where

$$B^T_{-\kappa} = \begin{bmatrix} \binom{0}{0}(-\kappa)^0 & \binom{1}{0}(-\kappa)^1 & \binom{2}{0}(-\kappa)^2 & \cdots & \binom{N}{0}(-\kappa)^N \\ 0 & \binom{1}{1}(-\kappa)^0 & \binom{2}{1}(-\kappa)^1 & \cdots & \binom{N}{1}(-\kappa)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\kappa)^0 & \cdots & \binom{N}{2}(-\kappa)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{2}(-\kappa)^0 \end{bmatrix}.$$

By using Equs. (3.26) - (3.28), we have:

$$\nu(\xi, \tau - \kappa) = G^T(\xi) U B^T_{-\kappa} H^T \ell(\tau). \tag{3.29}$$

3.6.2 Computation of (VFD) of $\nu(\xi,\tau)$:

Here, we expand $D_{\tau}^{\delta(\xi,\tau)}, 0 < \delta(\xi,\tau) \leq 1$ in terms of the (SℓPs), by using equ. (3.25), we get:

$$\begin{aligned} D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) &= I_{\tau}^{1-\delta(\xi,\tau)}(\frac{\partial \nu(\xi,\tau)}{\partial \tau}) \\ &\simeq \xi^2\tau^{2-\delta(\xi,\tau)}G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ \vartheta_2\ \hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau) + \xi I_{\tau}^{1-\delta(\xi,\tau)}(\acute{\nu}_1(\tau) - \acute{\nu}_0(\tau)) \\ &- \xi\tau^{2-\delta(\xi,\tau)}S^T D_1^T\vartheta_1^T U\vartheta_2\ \hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau) + \frac{\Gamma(1)}{\Gamma(2-\delta(\xi,\tau)}\tau^{1-\delta(\xi,\tau)}g_1(\xi) \\ &+ I_{\tau}^{1-\delta(\xi,\tau)}\acute{\nu}_0(\tau). \end{aligned} \tag{3.30}$$

So that

$$I_{\tau}^{1-\delta(\xi,\tau)}(\tau\ell(\tau)) \simeq \tau^{2-\delta(\xi,\tau)}\hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau),$$

since

$$\hat{\Theta}_N^{1-\delta(\xi,\tau)} = D_2\hat{\vartheta}_N^{1-\delta(\xi,\tau)}D_2^{-1}.$$

Also, for $1 < \delta(\xi,\tau) \leq 2$,

$$\begin{aligned} D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) &= I_{\tau}^{2-\delta(\xi,\tau)}(\frac{\partial^2 \nu(\xi,\tau)}{\partial \tau^2}) \\ &\simeq \xi^2\tau^{2-\delta(\xi,\tau)}G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ \vartheta_2\ \hat{\Theta}_N^{2-\delta(\xi,\tau)}\ell(\tau) + \xi I_{\tau}^{2-\delta(\xi,\tau)}(\acute{\acute{\nu}}_1(\tau) - \acute{\acute{\nu}}_0(\tau)) \\ &- \xi\tau^{2-\delta(\xi,\tau)}S^T D_1^T\vartheta_1^T U\vartheta_2\ \hat{\Theta}_N^{2-\delta(\xi,\tau)}\ell(\tau) + I_{\tau}^{2-\delta(\xi,\tau)}\acute{\acute{\nu}}_0(\tau). \end{aligned} \tag{3.31}$$

Substituting the approximations Equs. (3.17), (3.18) and (3.31) into Equ. (3.1) and the nodal points of Newton-Cotes, [74], then we get an algebraic system of equations and by using the Newtons iterative method. We get the unknown matrix U.
Substituting U into Equ. (3.20), we attain the approximate solution of the problem (3.1)-(3.3).

3.7 Convergence analysis of approximate solution

The next theorem tell us that the Shifted Legendre-Lagurre operational matrices can approximating an arbitrary continuous function [75]

3.7.1 Maximum error

We demonstrate uniform convergence of the Legendre-Laguerre expansion of the continuous function $\nu(\xi, \tau)$. Prior to that, however, we offer the upper bound for its error as follows:
Let $Q_{N,M}$ be a collection of all polynomials with maximum degrees of N for ξ and M for τ. Therefore, there exists a unique $q_{N,M} \in Q_{N,M}$ such that for $\nu \in C(\Omega)$.

$$\|\nu(\xi, \tau) - \nu_{N,M}(\xi, \tau)\| \leq \|\nu(\xi, \tau) - q_{N,M}(\xi, \tau)\|_{L^2_\omega(\Omega)}. \tag{3.32}$$

and we define

$$L^2_\omega(\Omega) = \{\Phi : \Phi \; is \; measurable \; on \; \Omega \; and \; \|\Phi\| < \infty\},$$

with inner product norm

$$\langle \Phi, \varrho \rangle = \int_\Omega \Phi(\xi, \tau)\varrho(\xi, \tau)\omega(\xi, \tau) d\xi d\tau,$$

Definition 3.1.

If $\nu(\xi, \tau)$ is a function with two variables and it's continuous at the point (ξ_0, τ_0) we have all its partial derivatives are also continuous at that point, then by Taylor series of $\nu(\xi, \tau)$ about the point (ξ_0, τ_0) it is calculated as:

$$\nu(\xi, \tau) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \frac{\partial^m}{\partial \tau^m} (\frac{\partial^n \nu}{\partial \xi^n})|_{(\xi_0, \tau_0)} (\xi - \xi_0)^n (\tau - \tau_0)^m$$

Also, we have

$$\nu(\xi, \tau) = \sum_{n=0}^{N} \sum_{m=0}^{M} \frac{1}{n!m!} \frac{\partial^m}{\partial \tau^m} (\frac{\partial^n \nu}{\partial \xi^n})|_{(\xi_0, \tau_0)} (\xi - \xi_0)^n (\tau - \tau_0)^m + R_{NM}(\xi, \tau). \tag{3.33}$$

As well as, where all partial derivatives of ν of order $N + M + 2$ belong to $L^2_\omega(\Omega)$, then

$$|R_{NM}(\xi, \tau)| \leq \frac{(\xi - \xi_0)^{N+1} (\tau - \tau_0)^{M+1}}{(N + 1)!(M + 1)!} \times sup_{(\xi, \tau) \in (\Omega)} |\frac{\partial^{N+M+2} \nu(\xi, \tau)}{\partial \xi^{N+1} \partial \tau^{M+1}}| \tag{3.34}$$

Theorem 3.1.

Assume that the real sufficiently smooth function ν, is expanded by the Legendre-Laguerre functions in Ω, as

$$\nu(\xi, \tau) = \sum_{n=0}^{N} \sum_{m=0}^{M} \widetilde{\nu}_{nm}(\xi, \tau) \Psi_{nm}(\xi, \tau) = \widetilde{V}^T \Psi_{nm}(\xi, \tau),$$

where

$$\Psi_{nm}(\xi,\tau) = [\Psi_{00}(\xi,\tau), \Psi_{01}(\xi,\tau), ..., \Psi_{0M}(\xi,\tau), ..., \Psi_{N0}(\xi,\tau), \Psi_{N2}(\xi,\tau), ..., \Psi_{NM}(\xi,\tau)]^T,$$

$$\widetilde{V} = [\widetilde{\nu}_{00}, \widetilde{\nu}_{01}, ..., \widetilde{\nu}_{0M}, ..., \widetilde{\nu}_{M0}, \widetilde{\nu}_{M1}, ..., \widetilde{\nu}_{NM}]$$

if the magnitude of the bounded on the right side of (3.34) by

$$C_{NM} = \ sup_{(\xi,\tau)\in(\Omega)} |\frac{\partial^{N+M+2}\nu(\xi,\tau)}{\partial\xi^{N+1}\partial\tau^{M+1}}|$$

The upper bound of error can be calculated as

$$\|\nu(\xi,\tau) - \nu_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)} \le \frac{C_{NM}\sqrt{(2M+2)!}}{(N+1)!(M+1)!\sqrt{(2N+3)}}. \tag{3.35}$$

Assume that

$$\widetilde{\nu}_{NM}(\xi,\tau) = \widetilde{V}^T\Psi_{NM}(\xi,\tau);$$

be the rough resolution obtained using the technique suggested in Section 3.6, where

$$\widetilde{V} = [\widetilde{\nu}_{00}, \widetilde{\nu}_{01}, ..., \widetilde{\nu}_{0M}, ..., \widetilde{\nu}_{M0}, \widetilde{\nu}_{M1}, ..., \widetilde{\nu}_{NM}].$$

Then

$$\|\nu(\xi,\tau) - \nu_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)} \le \frac{C_{NM}}{(N+1)!(M+1)!} \times \sqrt{\frac{(2M+2)!}{(2N+3)}} + \Theta_{NM}\|V - \widetilde{V}\|_2, \tag{3.36}$$

where

$$\Theta_{NM} = \sum_{n=0}^{N}\sqrt{\frac{M+1}{2n+1}},$$

and the norm $\|.\|_2$ is the standard Euclidean vector norm.

Proof: If we define

$$q_{NM}(\xi,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\frac{\partial^m}{\partial\tau^m}(\frac{\partial^n\nu}{\partial\xi^n})|_{(0,0)}\xi^n\tau^m, \tag{3.37}$$

from Eqs. (3.33) and (3.34) about $(\xi_0,\tau_0) = (0,0)$, we get

$$\|\nu(\xi,\tau) - q_{NM}(\xi,\tau)\| \le \frac{C_{NM}}{(N+1)!(M+1)!}\xi^{N+1}\tau^{M+1} \tag{3.38}$$

Applying the previous equation, we obtain

$$\begin{aligned}\|\nu(\xi,\tau)-\nu_{NM}(\xi,\tau)\|^2_{L^2_\omega(\Omega)} &= \int_0^\infty\int_0^1 |\nu(\xi,\tau)-\widetilde{V}^T\Psi_{nm}(\xi,\tau)|^2 e^{-\tau}d\xi d\tau\\ &\leq \int_0^\infty\int_0^1 |\nu(\xi,\tau)-q_{NM}(\xi,\tau)\Psi_{nm}(\xi,\tau)|^2 e^{-\tau}d\xi d\tau\\ &\leq \int_0^\infty\int_0^1 |\frac{C_{NM}}{(N+1)!(M+1)!}\xi^{N+1}\tau^{M+1}|^2 e^{-\tau}d\xi d\tau\\ &= C_{NM}^2\int_0^\infty\int_0^1 (\frac{\xi^{N+1}\tau^{M+1}}{(N+1)!(M+1)!})^2 e^{-\tau}d\xi d\tau\\ &= \frac{C_{NM}^2(2M+2)!}{((N+1)!(M+1)!)^2(2N+3)},\end{aligned} \tag{3.39}$$

The upper bound of the error is obtained by taking the square roots of both sides. And one can simply discover that

$$\|\nu(\xi,\tau)-\widetilde{\nu}_{NM}(\xi,\tau)\|^2_{L^2_\omega(\Omega)} \leq \|\nu(\xi,\tau)-\nu_{NM}(\xi,\tau)\|^2_{L^2_\omega(\Omega)}+\|\nu(\xi,\tau)-\widetilde{\nu}_{NM}(\xi,\tau)\|^2_{L^2_\omega(\Omega)}. \tag{3.40}$$

Next, we have

$$\begin{aligned}\|\nu_{NM}(\xi,\tau)-\widetilde{\nu}_{NM}(\xi,\tau)\|^2_{L^2_\omega(\Omega)} &= (\int_0^\infty\int_0^1 |\nu_{NM}(\xi,\tau)-\widetilde{\nu}_{NM}(\xi,\tau)|^2 e^{-\tau}d\xi d\tau)^{\frac{1}{2}}\\ &\leq (\int_0^\infty\int_0^1 |(V-\widetilde{V})\Psi_{NM}(\xi,\tau)|^2 e^{-\tau}d\xi d\tau)^{\frac{1}{2}}\\ &\leq (\int_0^\infty\int_0^1 [\sum_{n=0}^{N}\sum_{m=0}^{M}|V-\widetilde{V}|^2]\\ &\times [\sum_{n=0}^{N}\sum_{m=0}^{M}|\Psi_{NM}(\xi,\tau)|^2]e^{-\tau}d\xi d\tau)^{\frac{1}{2}}\\ &= (\sum_{n=0}^{N}\sum_{m=0}^{M}|V-\widetilde{V}|^2)^{\frac{1}{2}}\times(\sum_{n=0}^{N}\sum_{m=0}^{M}\int_0^\infty\int_0^1|\Psi_{NM}(\xi,\tau)|^2 e^{-\tau}d\xi d\tau)^{\frac{1}{2}}\\ &= \|V-\widetilde{V}\|_2(\sum_{n=0}^{N}\frac{M+1}{2m+1})^{\frac{1}{2}}\end{aligned} \tag{3.41}$$

As a result, from (2.10)- (3.41), we get

$$\begin{aligned}\|\nu(\xi,\tau)-\nu_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)} &\leq \frac{C_{NM}}{(N+1)!(M+1)!}\times\sqrt{\frac{(2M+2)!}{(2N+3)}}\\ &+\sum_{n=0}^{N}\sqrt{\frac{M+1}{2n+1}}\|V-\widetilde{V}\|_2\end{aligned}$$

According to the aforementioned theorem, the error tends to zero as the terms of Legendre-Laguerre functions increase.

3.8 Numerical Examples

To demonstrate the ability of the proposed method for solving (VFDDEs), two tested examples are given:

Example 3.1.

Consider the (VFDDEs) (3.1) with $\eta = 1, \kappa = 0.1$ and subject to:

$$\nu(0,\tau) = 0, \quad \nu(1,\tau) = 0, \quad \tau \in [0,\infty), \tag{3.42}$$

$$\nu(\xi,0) = 10\xi^2\ (1-\xi)^2, \qquad \frac{\partial \nu(\xi,0)}{\partial \tau} = 0, \tag{3.43}$$

where

$$f(\nu(\xi,\tau),\nu(\xi,\tau-\kappa)) = 10\xi^2(1-\xi)^2\frac{\tau^{2-\delta(\xi,\tau)}}{\Gamma(3-\delta(\xi,\tau))} - 20(6\xi^2-6\xi+1)(\tau^2+1)$$
$$- 10(\tau-0.1+1)^2\xi^2(1-\xi)^2.$$

This problem has an exact solution $\nu(\xi,\tau) = 10\xi^2(1-\xi)^2(\tau^2+1)$ and

$$\delta(\xi,\tau) = \frac{9}{5} - 0.005\ cos(\xi\tau)\ sin(\xi).$$

Figs.3.1 and 3.2 represent the absolute error of example 3.1 for M=N=8 and distinct values of $\delta(\xi,\tau)$. Also, Figs.3.3 and 3.4 represent a comparison between the exact solution and the approximate solution using the proposed method.

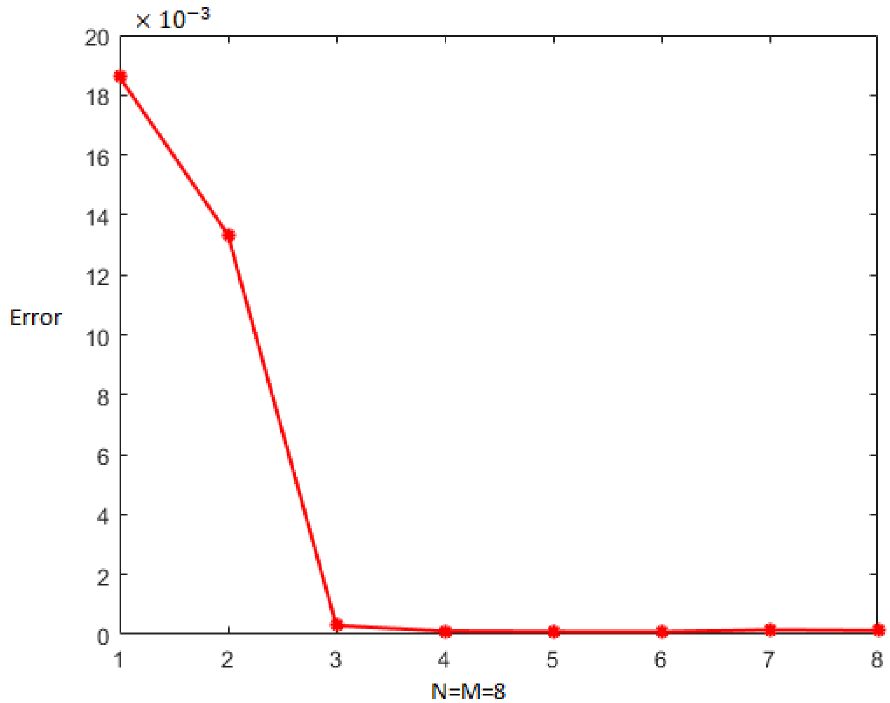

Figure 3.1: Absolute error absolute error when $\delta(\xi,\tau) = \frac{9}{5} - 0.005\ cos(\tau\xi)\ sin(\xi), \tau = \xi = 0.7$

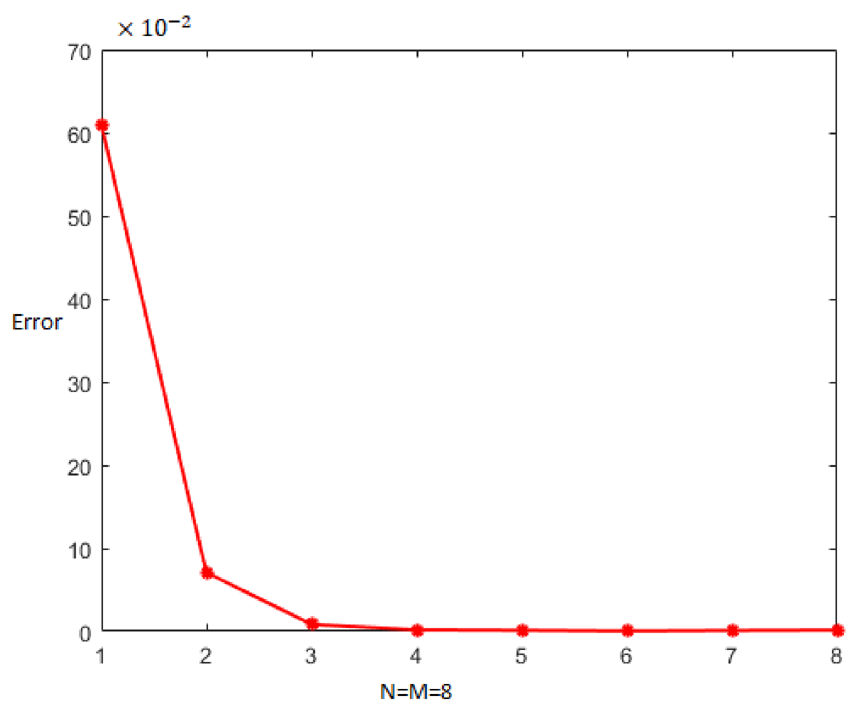

Figure 3.2: Absolute error when $\delta(\xi,\tau) = 1.7 + e^{-\xi\tau}, \tau = \xi = 0.8$

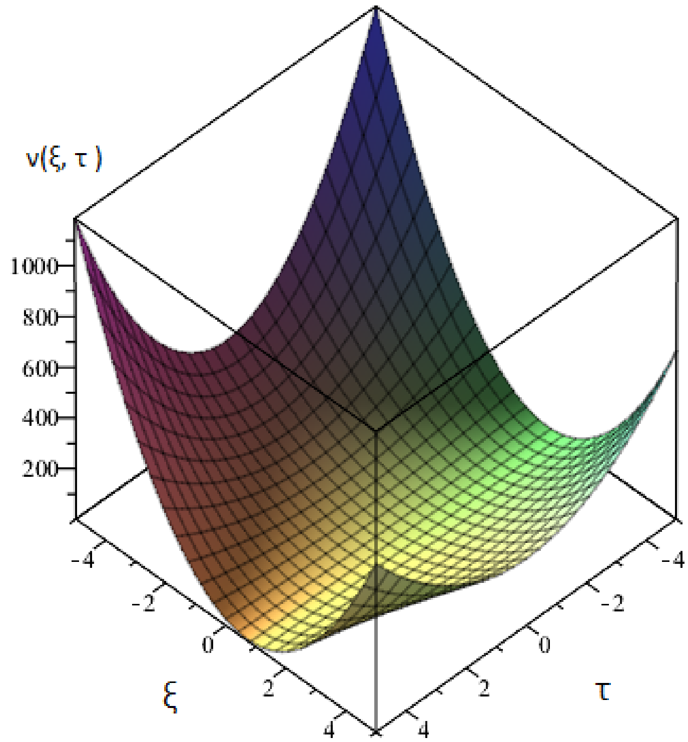

Figure 3.3: *Exact solution*

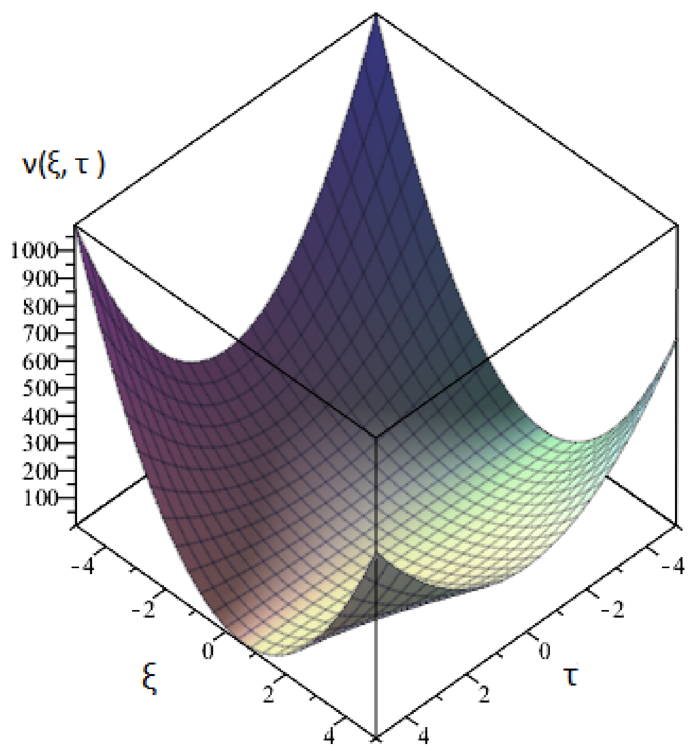

Figure 3.4: *Approximate solution*

Example 3.2.

Consider the VFDDEs (3.1) with $\eta = 1, \quad \kappa = 0.2$ and subject to:

$$\nu(0,\tau) = 0, \qquad \nu(1,\tau) = 0, \qquad \tau \in [0,\infty), \tag{3.44}$$

$$\nu(\xi,0) = \frac{\partial \nu(\xi,0)}{\partial \tau} = 5\xi\ (1-\xi), \tag{3.45}$$

where

$$f(\nu(\xi,\tau),\nu(\xi,\tau-\kappa)) = 5\xi(1-\xi)\frac{\tau^{1-\delta(\xi,\tau)}}{\Gamma(2-\delta(\xi,\tau))} - 10\tau + 5\xi(1-\xi)(\tau - 0.2 + 1).$$

This problem has a exact solution $\nu(\xi,\tau) = 5\xi(1-\xi)(\tau+1)$ and

$$\delta(\xi,\tau) = 2 - 0.2\ cos(\tau)\ sin(\xi).$$

Figs. 3.5 and 3.6 represent absolute error of example 3.2 for M=N=8 and distinct values of $\delta(\xi,\tau)$. A comparison between the exact and the approximate solutions of example 3.2 are given in Figs. 3.7 and 3.8.

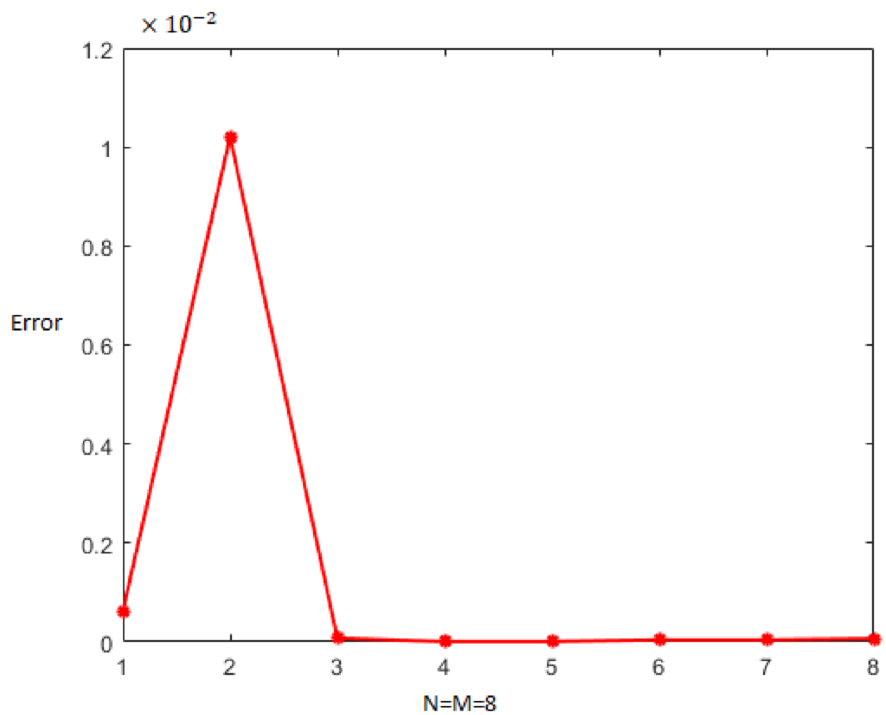

Figure 3.5: Absolute error when $\delta(\xi,\tau) = 2 - 0.2\ cos(\tau)sin(\xi), \tau = \xi = 0.7$

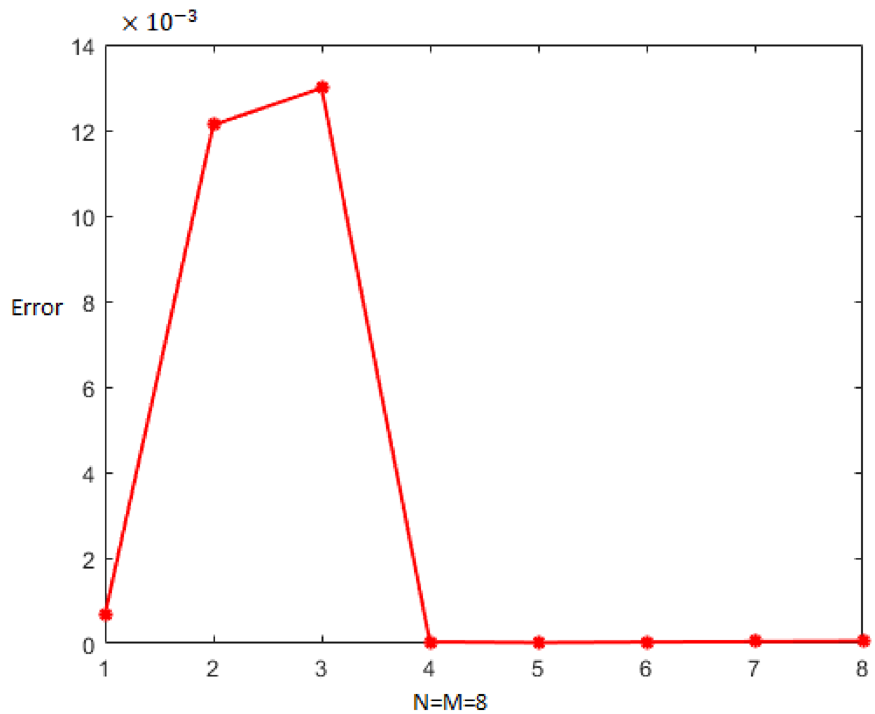

Figure 3.6: Absolute error when $\delta(\xi,\tau) = 1 + \frac{1}{2}sin(\tau\xi)exp(-\tau), \tau = \xi = 0.8$

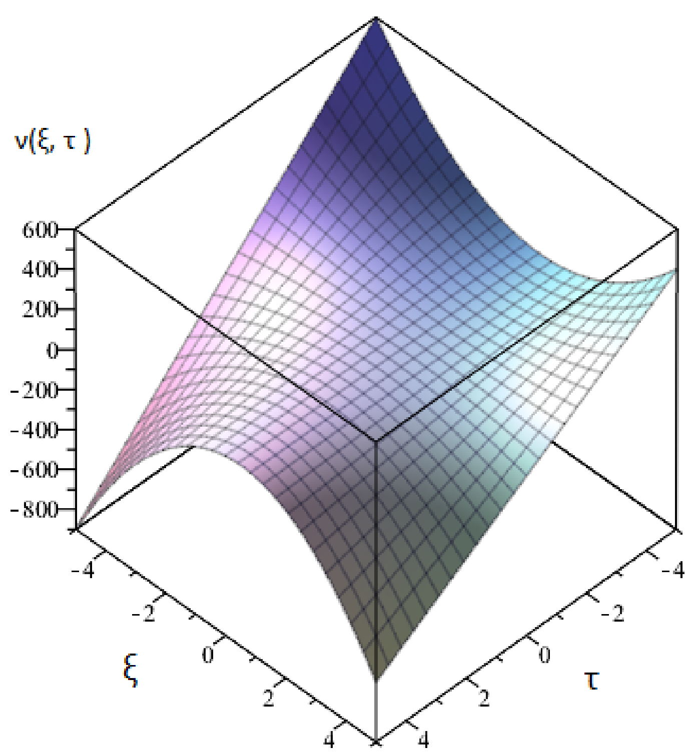

Figure 3.7: *Exact solution*

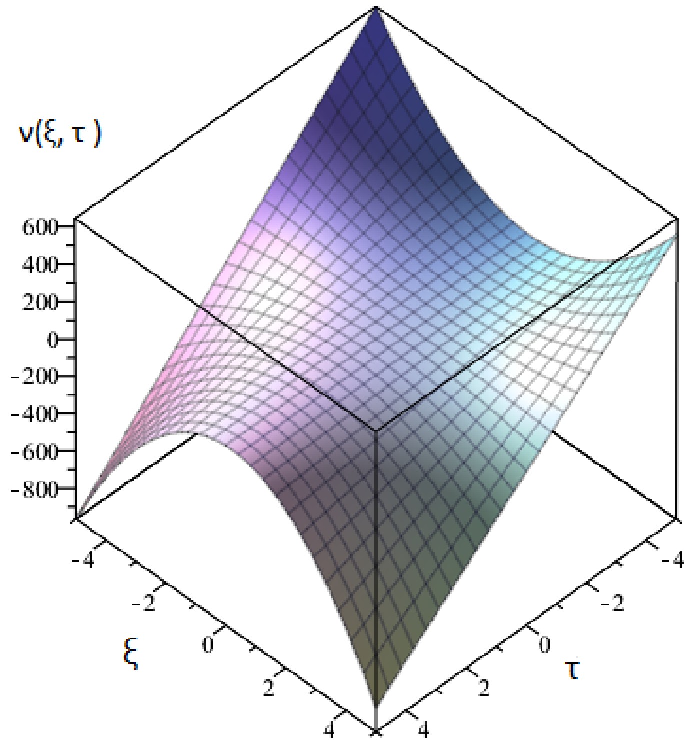

Figure 3.8: *Approximate solution*

Chapter 4

Homotopy perturbation method for solving time-fractional nonlinear Variable-Order Delay Partial Differential Equations

4.1 Introduction

The time-fractional nonlinear variable-order delay partial differential equations VOFDPDEs can be solved using the Homotopy perturbation method, which is a potent numerical approach. A time-fractional partial differential equation VOFDPDEs depicts the development of a time-dependent field in which the quantity's diffusion is influenced by both the field's history and its spatial gradient. Traditional approaches may have trouble solving this problem because of the variable-order derivative. By creating a Homotopy, which is a continuous deformation of the equation of interest into a smaller equation that can be readily solved, the Homotopy perturbation method achieves its desired results. The simpler equation's solution is then found using a perturbation approach, and the old equation is gradually bent back into it. As a result, the original equation's exact solution is reached after a succession of approximations.The time-fractional VODE can be transformed into a set of algebraic equations using the Homotopy perturbation method, which can then be solved numerically. High precision, minimal computing expense, and support for nonlinear and nonlocal problems with variable-order derivatives are only a few benefits of this approach. Overall, the Homotopy perturbation method offers a viable strategy for resolving time-fractional nonlinear VOFDPDEs with variable-order derivatives and may find significant use in physics, engineering, and biology, among other disciplines.

4.2 Homotopy perturbation method

Professor J.H. He first introduced the Homotopy Perturbation Method HPM in 1999. The methodology was created as a quick, effective, and potent tool for dealing with nonlinear issues that are frequently challenging or impossible to address using traditional analytical methods. Professor In a paper titled "Homotopy Perturbation Technique" in the Journal of Applied Mathematics and Computing, he initially described his technique. The method's fundamental ideas were detailed in the paper, along with examples of how it could be applied to resolve a variety of nonlinear issues, including certain differential equations and integral equations.Since its creation, the HPM has seen extensive use in a variety of nonlinear issues across several industries, including engineering, physics, and applied mathematics. Systems with numerous degrees of freedom and issues involving differential and integral equations have both been successfully solved using this approach. The method has seen numerous changes and advancements throughout time, including the use of various homotopy functions and perturbation strategies. The HPM is still one of the most effective and versatile tools for dealing with nonlinear issues, and there is still a lot of research being done in this area of computational science and mathematical modeling. The basic consider of Homotopy perturbation method illustrated by consider the following nonlinear functional equation

$$A(u) = y(\xi) \tag{4.1}$$

with the boundary conditions, $(U, \frac{\partial u}{\partial n}) = 0$, $\xi \in \Gamma$, where A is a general functional operator, U is a boundary operator, $y(\xi)$ is a known analytic function, and Γ is the boundary of domain Ω, the operator A can be decomposed into two parts L and N, where L is linear and N is a nonlinear operator, equation 4.1 can by rewritten as the following:

$$L(u) + N(u) - y(\xi) = 0. \tag{4.2}$$

We construct a Homotopy $U(x,p) : \Omega \times [0,1] \longrightarrow R$, which satisfies:

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[A(U) - y(\xi)] = 0, \tag{4.3}$$

where $p \in [0,1]$, $\xi \in \Omega$ or

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - y(\xi)] = 0, \tag{4.4}$$

where u_0 is an initial approximation for the solution of equation 4.1. In this method, we use the Homotopy parameter p to expand the approximate will be obtained by taking the limit as p tends to 1,

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + \dots \quad . \tag{4.5}$$

4.3 Fractional partial differential equation with delay

As an example, think about equation for a delay-fractional partial differential

$$D_\tau^\delta \nu(\xi,\tau) - K\frac{\partial^2}{\partial \xi^2}\nu(\xi,\tau) = F(\xi,\tau,\nu(\xi,\tau),\nu(\xi,\tau-\kappa)) \tag{4.6}$$

where $0 < \delta < 1$, K is a constant, F is a nonlinear function, and $\kappa > 0$ is the time delay.

We begin by building the Homotopy as follows in order to solve this equation utilizing the Homotopy perturbation method:

$$\begin{aligned} H(\xi,\tau,p,q,u,\lambda) = D_\tau^\delta \nu(\xi,\tau) - K\frac{\partial^2}{\partial x^2}\nu(\xi,\tau) - \lambda(f(\xi,\tau,\nu(\xi,\tau),\nu(\xi,\tau-\tau)) - u) \\ + \lambda q\nu(\xi,\tau) - pF(\xi,\tau,\nu(\xi,\tau),\nu(\xi,\tau-\kappa)), \end{aligned} \tag{4.7}$$

where p and q are auxiliary functions, u is a constant, and λ is the Homotopy parameter.

Next, we assume a solution of the form:

$$\nu(\xi,\tau) = \sum_{n=0}^{\infty} \lambda^n \nu_n(\xi,\tau). \tag{4.8}$$

Substituting this expression into the Homotopy and equating the coefficients of different powers of λ, we can derive a recursive formula for the solution. After some algebraic manipulation, we arrive at the following formula for the nth approximation of the solution:

$$\nu_n(\xi,\tau) = \frac{1}{\Gamma(n+\gamma)}\int_0^\tau (\tau-s)^{\gamma-1} e^{\frac{K}{2}(\xi^2-y^2)} F(\xi,s,\nu_{n-1}(\xi,s),\nu_{n-1}(\xi,s-\kappa))ds, \tag{4.9}$$

where Γ is the gamma function, and $y = (\tau-s)^{1-\gamma}\xi + s^{1-\gamma}y$. We can find a rough solution to the delay fractional partial differential equation by iterating this formula. It's important to note that the selection of the auxiliary functions p and q, which can be done by utilizing optimization techniques or by employing trial and error, determines the correctness and convergence of the solution.

4.4 Problem statement

The propagation of waves in shallow water is described by the Korteweg-de Vries KdV equation, a nonlinear partial differential equation. It was created in 1895 as a model for lengthy waves in a canal with varied depths by Diederik Korteweg and Gustav de Vries. However, the KdV equation did not become well known as a fundamental model in the study of nonlinear waves until the middle of the 20th century. The study and solution of the KdV equation have a long and intriguing history. Early researchers had trouble coming up with analytical answers to the equation, and it wasn't until the 1960s that V. E. Zakharov and A.B. Shabat discovered a breakthrough. They were able to demonstrate that the KdV equation is integrable, indicating that inverse scattering methods can be used to solve it. This finding led to further developments in the understanding and solution of nonlinear partial differential equations and opened up a completely new field of study in the area of integrable systems. Solitons theory, spectral theory, and symmetry analysis are a few of the techniques that researchers have developed over time to solve the KdV problem.The KdV equation is still useful today for studying nonlinear wave events, and its solutions have uses in nonlinear optics, plasma physics, and fluid dynamics, among other disciplines. In this chapter will investigate the Korteweg-de Vries KdV equation's solution.

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \mathcal{F}\left(\xi,\tau,\nu(p_0\xi,q_0\tau),\frac{\partial}{\partial\xi}\nu(p_1\xi,q_1\tau),...,\frac{\partial^n}{\partial\xi^n}\nu(p_n\xi,q_n\tau)\right), \quad (4.10) $$

with the initial condition,

$$ \nu^{\kappa}(\xi,0) = g_{\kappa}(\xi), \quad (4.11) $$

where p_i, q_j belong in the interval $(0,1)$ for $i,j \in N, g_{\kappa}(\xi)$ is a specific starting values, and $\mathcal{F}$ is the differential effect with partial derivatives.

4.5 Application of (HPM) for Solving (VFPDDEs)

We review a study time-fractional variable order DPDEs:

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \mathcal{F}\left(\xi,\tau,\nu(p_0\xi,q_0\tau),\frac{\partial}{\partial\vartheta}\nu(p_1\xi,q_1\tau),...,\frac{\partial^n}{\partial\xi^n}\nu(p_n\xi,q_n\tau)\right), \quad (4.12) $$

where, $\xi,\tau \in [0,1]$ and ${}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}$ is the fractional derivative of order $\delta(\xi,\tau)$ in relation to $\tau, m-1 < \delta(\xi,\tau) \leq m, m \in N^{+}, \nu^{\kappa}(\xi,0) = g(\xi), \kappa = 0,1,2,...,p_i,q_j \in (0,1)$, for $i,j \in N, g(\xi)$ are specific starting values, and $\mathcal{F}$ is partial differential function. Homotopy perturbation theory states with respect to the method presented, we evaluated the following HPM of Eqs. (4.10) and (4.11)

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = H \left(F \left(\xi, \tau, \nu(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu(p_n\xi, q_n\tau) \right) \right). \tag{4.13} $$

Substituting Eq. (4.12) and the elementary conditions into the Homotopy Eq. (4.13) and equating the terms with identical powers of H , We have evaluate the following differential equations:

$$ H^0 : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_0(\xi,\tau) = F(\xi,\tau), $$

$$ H^1 : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_1(\xi,\tau) = F \left(\xi, \tau, \nu_0(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_0(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_0(p_n\xi, q_n\tau) \right), $$

$$ H^2 : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_2(\xi,\tau) = F \left(\xi, \tau, \nu_1(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_1(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_1(p_n\xi, q_n\tau) \right), $$

$$ H^3 : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_2(\xi,\tau) = F \left(\xi, \tau, \nu_2(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_2(p_1\xi, q_2\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_2(p_n\xi, q_n\tau) \right), $$

$$ \vdots $$

Operating $I_\tau^{\delta(\xi,\tau)}$ on the equations yields:

$$ \nu_0(\xi,\tau) = \sum_{k=0}^{m-1} \frac{\tau^k}{k!} \nu^\kappa(\xi,0) + I_\tau^{\delta(\xi,\tau)} F(\xi,\tau), $$

$$ \nu_1(\xi,\tau) = I_\tau^{\delta(\xi,\tau)} \left(F \left(\xi, \tau, \nu_0(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_0(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_0(p_n\xi, q_n\tau) \right) \right), $$

$$ \nu_2(\xi,\tau) = I_\tau^{\delta(\xi,\tau)} \left(F \left(\xi, \tau, \nu_1(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_1(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_1(p_n\xi, q_n\tau) \right) \right), $$

$$ \nu_3(\xi,\tau) = I_\tau^{\delta(\xi,\tau)} \left(F \left(\xi, \tau, \nu_2(p_0\xi, q_0\tau), \frac{\partial}{\partial \xi}\nu_2(p_1\xi, q_1\tau), ..., \frac{\partial^n}{\partial \xi^n}\nu_2(p_n\xi, q_n\tau) \right) \right), $$

$$ \vdots $$

where $\nu_0(\xi,\tau)$ is an initial approximation for the solution of Eq.(4.12). The solution of Eq.(4.12) can be decomposed as a power series in H :

$$ \nu(\xi,\tau) = \nu_0(\xi,\tau) + H\nu_1(\xi,\tau) + H^2\nu_2(\xi,\tau) + H^3\nu_3(\xi,\tau) + \dots \tag{4.14} $$

Setting $H = 1$ in Eq.(4.14), gives the approximate solution of Eq.(4.12),as:

$$\nu(\xi,\tau) = \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + \nu_3(\xi,\tau) + \dots \tag{4.15}$$

4.6 Analysis of convergence and estimation of error

In this section, we focus on the convergence of the HPM for Eqs.(4.10). The sufficient conditions for convergence of the method and the error estimate are presented [76, 77].

Theorem 4.1.

Let $\nu_m(\xi,\tau)$ and $\nu(\xi,\tau)$ be defined in Banach space $(C[0,1], \|.\|)$. Then the series solution $\{\nu_m(\xi,\tau)\}_{m=0}^{\infty}$ defined by Eq.(4.15) converges to the solution of Eq.(4.10).

Proof

Soppose that $(C[0,1], \|.\|)$. is the Banach space of all continuing functions on $[0,1]$ backed by norm $\| \nu(\xi,\tau) \| = max_{\forall \xi,\tau\in[0,1]} \mid \nu(\xi,\tau) \mid$

Let A_n is the sequence of partial sums of the series Eq. (4.15) as,

$$\begin{cases} A_0(x) = \nu_0(\xi,\tau), \\ A_1(x) = \nu_0(\xi,\tau) + \nu_1(\xi,\tau), \\ A_2(x) = \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau), \\ \vdots \\ A_n(x) = \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + ... + \nu_n(\xi,\tau). \end{cases} \tag{4.16}$$

and we need to show that $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space $(C[0,1], \|.\|)$.For this purpose, we consider,

$$\begin{aligned} \| A_{n+1} - A_n \| = \| \nu_{n+1}(\xi,\tau) \| \leq \beta \| \nu_n(\xi,\tau) \| \leq \beta^2 \| \nu_{n-1}(\xi,\tau) \| \leq \\ \cdots \leq \beta^{n+1} \| \nu_0(\xi,\tau) \| . \end{aligned} \tag{4.17}$$

$\forall n, m \in N, n \geq m$, by using Eq.(4.17)and triangle inequality successively, we get,

$$\begin{aligned}\| A_n - A_m \| &= \| (A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + ... + (A_{m+1} - A_m) \| \\ &\leq \| (A_n - A_{n-1}) \| + \| (A_{n-1} - A_{n-2}) \| +...+ \| (A_{m+1} - A_m) \| \\ &\leq \beta^n \| \nu_0(\xi, \tau) \| + \beta^{n-1} \| \nu_0(\xi, \tau) \| +... + \beta^{m+1} \| \nu_0(\xi, \tau) \| \\ &= \frac{1 - \beta^{n-m}}{1 - \beta} \beta^{m+1} \| \nu_0(\xi, \tau) \| .\end{aligned} \tag{4.18}$$

since $0 < \beta < 1$, then $1 - \beta^{n-m} < 1 \Longrightarrow$

$$\| A_n - A_m \| \leq \frac{\beta^{m+1}}{1 - \beta} max_{\forall x, t \in [0,1]} \| \nu_0(\xi, \tau) \| . \tag{4.19}$$

since $\nu_0(\xi, \tau)$ is bounded, therefor

$$\lim_{n,m \longrightarrow \infty} \| A_n - A_m \| = 0 \tag{4.20}$$

Therefore, $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[0, 1], \| \, . \, \|)$, so the series solution defined in Eq. (4.15) , converges.

Theorem 4.2.

The maximum absolute truncation error of the series solution Eq.(4.15) for Eq.(4.10) is estimated to be

$$| \nu(\xi, \tau) - \sum_{i=0}^{m} \nu_i(\xi, \tau) | \leq \frac{\beta^{m+1}}{(1 - \beta)} \| \nu_0(\xi, \tau) \| . \tag{4.21}$$

where $0 < \beta < 1$.

Proof: By Theorem 4.2 and Eq.(4.18) we have

$$\| A_n - A_m \| = \frac{1 - \beta^{n-m}}{1 - \beta} \beta^{m+1} \| \nu_0(\xi, \tau) \| \tag{4.22}$$

for $n \geq m$. if $n \to \infty$ then $A_n \to \nu(\xi, \tau)$. So,

$$| \nu(\xi, \tau) - A_m | \leq \frac{\beta^{m+1}}{(1 - \beta)} \| \nu_0(\xi, \tau) \| . \tag{4.23}$$

Since $0 < \beta < 1$, we get $1 - \beta^{n-m} < 1$. Then above inequality becomes,

$$| \nu(\xi,\tau) - \sum_{i=0}^{m} \nu_i(\xi,\tau) | \leq \frac{\beta^{m+1}}{(1-\beta)} \parallel \nu_0(\xi,\tau) \parallel . \tag{4.24}$$

4.7 Illustrative Examples

The HPM described in the preceding section will be utilized to some (VF-PDDEs).

Example 4.1.

Think about the solution of the following proportional delay generalized time-fractional Burgers equation:

$${}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \nu(\xi, \frac{\tau}{3})\ \nu(\xi, \frac{\tau}{3}) - \frac{1}{3}\nu(\xi,\tau), \tag{4.25}$$

$\xi, \tau \in [0,1]$ and $\delta(\xi,\tau) = 2 - 0.3 sin(\xi\ \tau)$ for initial conditions $\nu(\xi, 0) = \xi^2$. The exact solution of this equation is $\nu(\xi,\tau) = \xi^2 cosh(1.3\tau)$. By HPM for Eq. (4.25) can be given as,

$${}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = H \left(\frac{\partial^2}{\partial \xi^2} \nu(\xi, \frac{\tau}{3})\ \nu(\tau, \frac{\tau}{3}) - \frac{1}{3}\nu(\xi,\tau) \right) \tag{4.26}$$

According to section 4.5, we construct the following set of (LDEs).

$H^0\ :\ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_0(\xi,\tau) = 0,$

$H^1\ :\ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_1(\xi,\tau) = \nu_{0,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_0(\xi, \frac{\tau}{3}) - \frac{1}{3}\nu_0(\xi,\tau),$

$H^2\ :\ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_2(\xi,\tau) = \nu_{0,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_1(\xi, \frac{\tau}{3}) + \nu_{1,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_0(\xi, \frac{\tau}{3}) - \nu_1(\xi,\tau),$

$$H^3\ :\ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_3(\xi,\tau) = \nu_{2,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_0(\xi, \frac{\tau}{3}) + \nu_{0,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_2(\xi, \frac{\tau}{3})\nu_1(\xi,\tau) + \nu_{1,\xi\ \xi}(\xi, \frac{\tau}{3})\nu_1(\xi, \frac{\tau}{3}) - \nu_2(\xi,\tau),$$

$\vdots$

The initial parts of the homotopy perturbation solution for Eq. (4.25) are thus derived by solving the aforementioned equations as follows:

$\nu_0(\xi,\tau) = \xi^2,$

$\nu_1(\xi,\tau) = a\ \xi^2 \tau^{\delta(\xi,\tau)},$

$$\nu_2(\xi,\tau) = (b-c)\,\xi^2\tau^{2\delta(\xi,\tau)},$$

$$\nu_3(\xi,\tau) = \left((b-c)d + 4ad3^{-2\delta(\xi,\tau)}\right)\xi^2\tau^{3\delta(\xi,\tau)},$$

$$\vdots$$

where

$$a = \frac{5}{3\,\Gamma\left(\delta(\xi,\tau)+1\right)},\quad b = \frac{20\times 3^{-1-\delta(\xi,\tau)}}{\Gamma\left(2\delta(\xi,\tau)+1\right)},\quad c = \frac{5}{3\,\Gamma\left(\delta(\xi,\tau)+1\right)},\quad d = \frac{\Gamma\left(2\,\delta(\xi,\tau)+1\right)}{\Gamma\left(3\,\delta(\xi,\tau)+1\right)}$$

The m^{th}-order approximate solution of Eq. (4.25)

$$\nu(\xi,\tau) \simeq \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + \nu_3(\xi,\tau) + ... \tag{4.27}$$

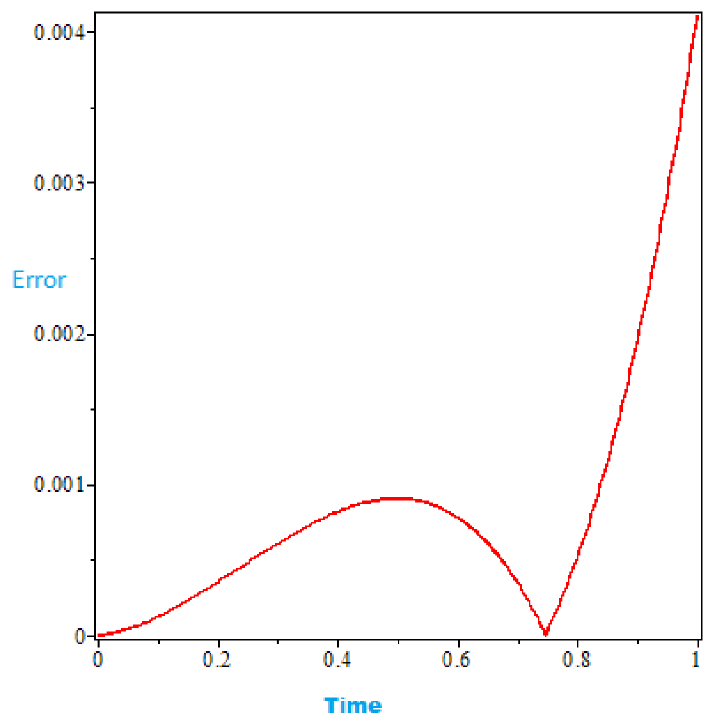

Figure 4.1: Absolute error of example 4.1 at $\delta(\xi,\tau) = 2 - 0.2sin(\xi\ \tau)$

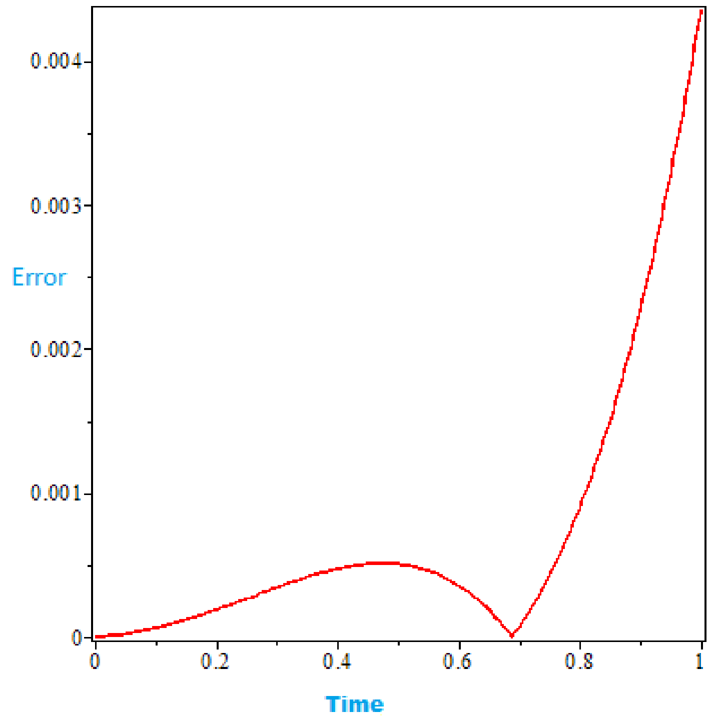

Figure 4.2: Absolute error of example 4.1 at $\delta(\xi,\tau) = 1.7 + 0.1e^{(-\xi\ \tau)}, \xi = 0.8$

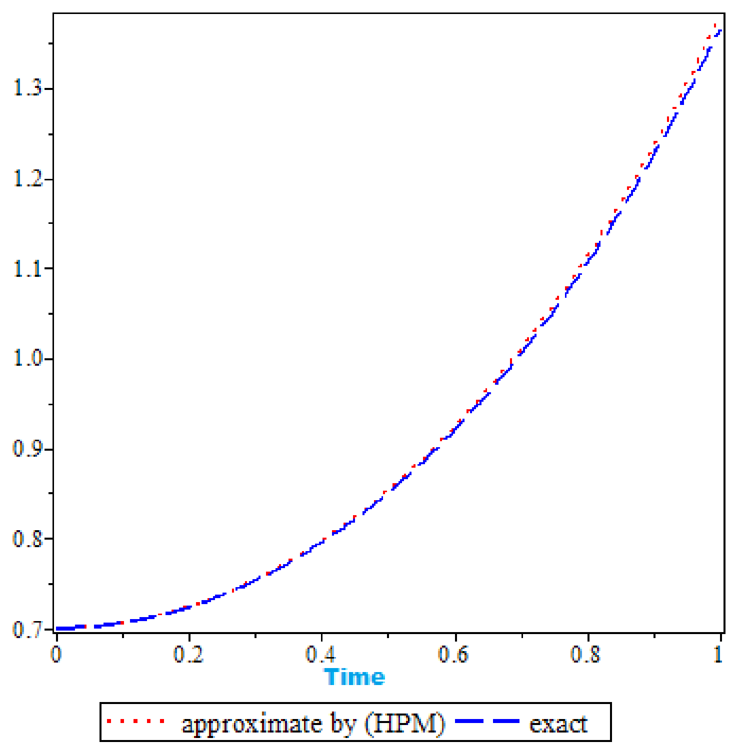

Figure 4.3: Solution of example 4.1 when $\delta(\xi,\tau) = 2 - 0.3sin(\xi\ \tau), \xi = 0.6$

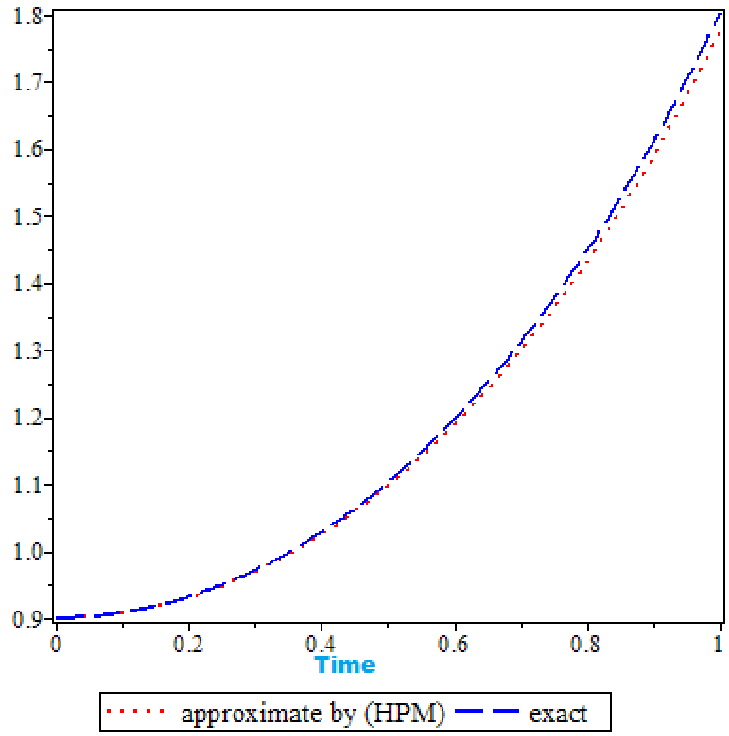

Figure 4.4: Solution of example 4.1 when $\delta(\xi, \tau) = 1.7 + 0.1e^{(-\xi\ \tau)}, \xi = 0.8$

Table 4.1: Absolute error for example 4.1 at $\delta(\xi, \tau) = 2 - 0.2sin(\xi\ \tau)$

(ξ, τ)	$AbsoluteError$
$(0.25, 0.25)$	5.865×10^{-4}
$(0.25, 0.50)$	1.775×10^{-4}
$(0.25, 0.75)$	1.903×10^{-4}
$(0.50, 0.25)$	8.265×10^{-3}
$(0.50, 0.50)$	3.181×10^{-4}
$(0.50, 0.75)$	9.717×10^{-4}
$(0.75, 0.25)$	3.233×10^{-4}
$(0.75, 0.50)$	1.458×10^{-3}
$(0.75, 0.75)$	1.903×10^{-3}

Figures 4.1 - 4.4 represent the absolute error and the approximate solution of Example 4.1 for $N = 3$ and different values of $\delta(\xi, \tau)$ at $\xi = 0.9$ respectively. Finally the absolute error for different values of (ξ, τ) is given in Table 4.1.

Example 4.2.

Consider the following proportional delay generalized time-fractional Burgers equation:

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \frac{\partial^2}{\partial\xi^2}\nu(\xi,\tau) + \nu(\xi,\frac{\tau}{2})\nu(\frac{\xi}{2},\frac{\tau}{2}) + \frac{1}{2}\nu(\xi,\tau), \tag{4.28} $$

with initial conditions $\nu(\xi,0) = \xi,\ \delta(\xi,\tau) = 1.7+0.3cos^2(\xi\ \tau)$. The exact solution of this problem is $\nu(\xi,\tau) = \xi e^{0.5\tau}$. By the HPM for Eq. (4.28) can be establish as,

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = H\left(\frac{\partial^2}{\partial\xi^2}\nu(\frac{\xi}{2},\frac{\tau}{2})\frac{\partial}{\partial\xi}\nu(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\frac{\partial}{\partial\xi}\nu(\xi,\tau) - \nu(\xi,\tau)\right), \tag{4.29} $$

Similarly we have:

$$
\begin{aligned}
H^0 &: {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu_0(\xi,\tau) = 0,\\
H^1 &: {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu_1(\xi,\tau) = \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\nu_{0,\xi}(\xi,\tau) - \nu_0(\xi,\tau),\\
H^2 &: {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu_2(\xi,\tau) = \nu_{1,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\xi,\frac{\tau}{2}) + \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{1,\xi}(\xi,\frac{\tau}{2}) - \frac{1}{8}\nu_{1,\xi}(\xi,\tau) - \nu_1(\xi,\tau),\\
H^3 &: {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu_3(\xi,\tau) = \nu_{2,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\xi,\frac{\tau}{2}) + \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{2,\xi}(\frac{\xi}{2},\frac{\tau}{2}) + \nu_{1,\xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{1,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\\
&\qquad - \frac{1}{8}\nu_{2,\xi}(\xi,\tau) - \nu_2(\xi,\tau),\\
&\vdots
\end{aligned}
$$

The initial parts of the Homotopy perturbation solution for Eq. (4.28) are thus derived by solving the aforementioned equations as follows:

$$
\begin{aligned}
\nu_0(\xi,\tau) &= \xi^2,\\
\nu_1(\xi,\tau) &= a\ \xi^2\tau^{\delta(\xi,\tau)},\\
\nu_2(\xi,\tau) &= 2\ a\ b\ \xi\ \tau^{2\delta(\xi,\tau)} - \frac{1}{4}\ b\ \xi\ \tau^{2\delta(\xi,\tau)} + c\ \xi\ \tau^{2\delta(\xi,\tau)},\\
\nu_3(\xi,\tau) &= \frac{\Gamma(2\delta(\xi,\tau)+1)}{\Gamma(3\delta(\xi,\tau)+1)}\tau^{2\delta(\xi,\tau)}[(c\ 2^{-}2\delta(\xi,\tau)\xi + 2\ a\ b2^{-1-2\delta(\xi,\tau)} - \frac{1}{4}\ b\ 2^{-1-2\delta(\xi,\tau)} + a^2 2^{-2\delta(\xi,\tau)}\xi\\
&\qquad - \frac{1}{4}ab + \frac{1}{16}b - \frac{1}{4}c\xi - 2ab\xi + \frac{1}{4}b\xi - c\xi^2)],\\
&\vdots
\end{aligned}
$$

where

$$a = \frac{-1}{\Gamma(\delta(\xi,\tau)+1)}, \qquad b = \frac{\Gamma(\delta(\xi,\tau)+1)}{\Gamma(2\delta(\xi,\tau)+1)} \qquad c = \frac{1}{2\Gamma(\delta(\xi,\tau)+1)} \tag{4.30}$$

The rest parts of the Homotopy perturbation solution can be produced in the same way for the subsequent components. The following is the approximate mth-order solution of Eq. ((4.28)):

$$\nu(\xi,\tau) \simeq \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + \nu_3(\xi,\tau) + ... \tag{4.31}$$

Figures 4.5 - 4.8 represent absolute error and the approximate solution of example 4.2 for different values of $\delta(\xi,\tau)$ at $\xi = 0.7$. Finally the absolute error for different value of (ξ,τ) is given in table 4.2.

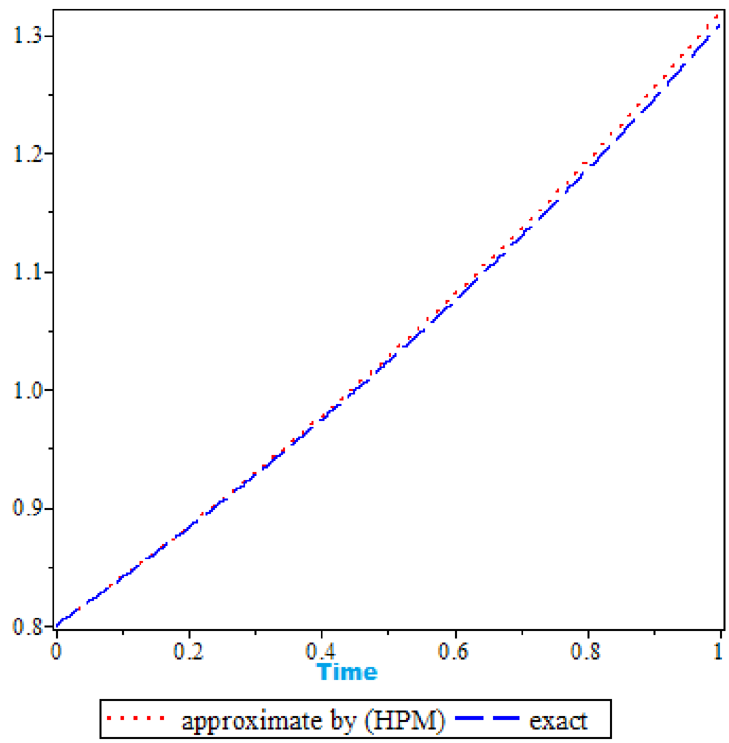

Figure 4.5: Solution of example 4.2 when $\delta(\xi,\tau) = 1.7 + 0.3cos^2(\xi\ \tau), \xi = 0.7$

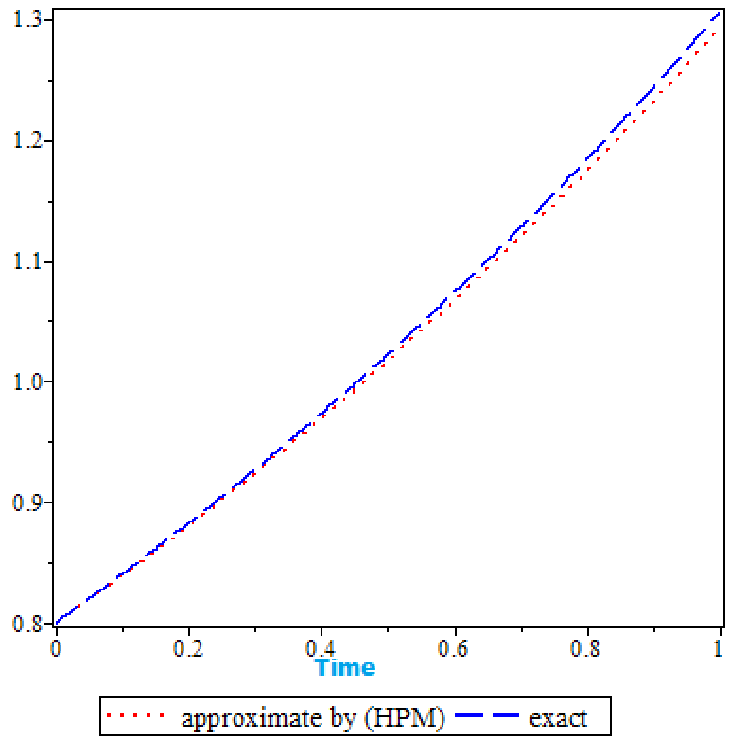

Figure 4.6: Solution of example 4.2 when $\delta(\xi,\tau) = 1.7 + 0.3cos^2(\xi\ \tau), \xi = 0.8$

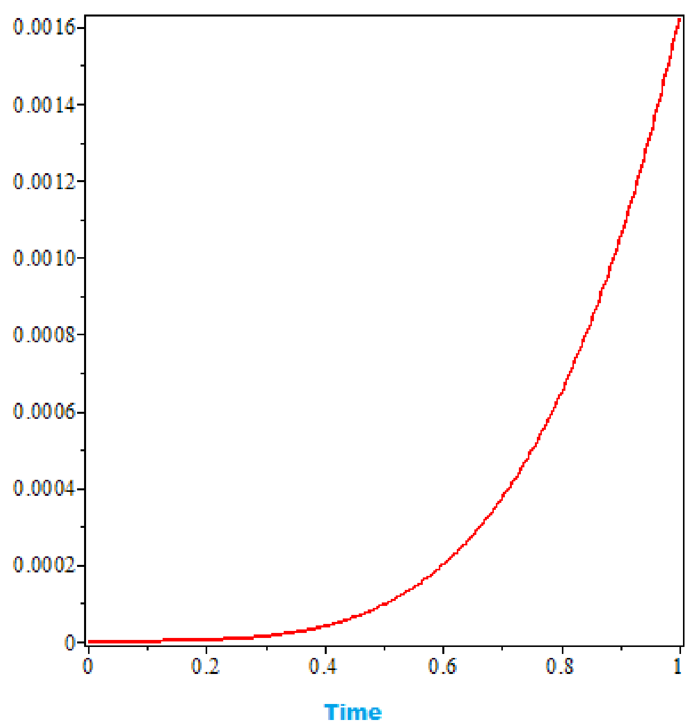

Figure 4.7: Absolute error of example 4.2 $at\ \xi = 0.7\ when\ \delta(\xi,\tau) = 1.7 + 0.3cos^2(\xi\ \tau), \xi = 0.8$

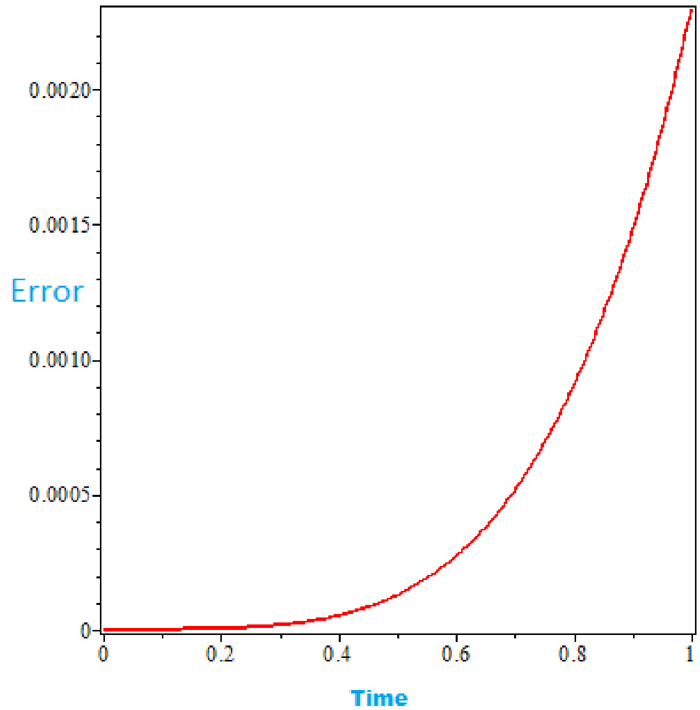

Figure 4.8: Absolute error of example 4.2 *at* $\xi = 0.7$ *when* $\delta(\xi,\tau) = 2 + 0.2sin(-\xi\ \tau), \xi = 0.6$

Table 4.2: Absolute error for example 4.2 at $\delta(\xi,\tau) = 1.7 + 0.3cos^2(\xi\ \tau)$

(ξ,τ)	$AbsoluteError$
$(0.25, 0.25)$	8.682×10^{-4}
$(0.25, 0.50)$	1.549×10^{-4}
$(0.25, 0.75)$	8.689×10^{-3}
$(0.50, 0.25)$	4.608×10^{-3}
$(0.50, 0.50)$	1.654×10^{-4}
$(0.50, 0.75)$	1.179×10^{-3}
$(0.75, 0.25)$	2.496×10^{-4}
$(0.75, 0.50)$	1.072×10^{-4}
$(0.75, 0.75)$	4.482×10^{-4}

Example 4.3. *Given the following FPDE with proportional delay:*

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \nu(\frac{\xi}{2}, \frac{\tau}{2}) \frac{\partial}{\partial \xi} \nu(\frac{\xi}{2}, \frac{\tau}{2}) - \frac{1}{8} \frac{\partial}{\partial \xi} \nu(\xi,\tau) - \nu(\xi,\tau), \tag{4.32} $$

$\xi, \tau \in [0,1]$ and $\delta(\xi,\tau) = 2 - 0.3\ exp(\xi\ \tau)$ with elementary conditions $\nu(\xi,0) = \xi^2$.

The exact solution of this problem is $\nu(\xi,\tau) = \xi^2 cos(\tau)$. By the HPM for Eq.

(4.32) can be establish as,

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = H\left(\frac{\partial^2}{\partial \xi^2}\nu(\frac{\xi}{2},\frac{\tau}{2})\frac{\partial}{\partial \xi}\nu(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\frac{\partial}{\partial \xi}\nu(\xi,\tau) - \nu(\xi,\tau)\right), \tag{4.33} $$

Also, as illustrated in section 3, we have the following set of differential equations:

$$ H^0 \ : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_0(\xi,\tau) = 0, $$

$$ H^1 \ : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_1(\xi,\tau) = \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\nu_{0,\xi}(\xi,\tau) - \nu_0(\xi,\tau), $$

$$ H^2 \ : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_2(\xi,\tau) = \nu_{1,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\xi,\frac{\tau}{2}) + \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{1,\xi}(\xi,\frac{\tau}{2}) - \frac{1}{8}\nu_{1,\xi}(\xi,\tau) - \nu_1(\xi,\tau), $$

$$ \begin{aligned} H^3 \ : \ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu_3(\xi,\tau) =& \nu_{2,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{0,\xi}(\xi,\frac{\tau}{2}) + \nu_{0,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{2,\xi}(\frac{\xi}{2},\frac{\tau}{2}) + \nu_{1,\xi}(\frac{\xi}{2},\frac{\tau}{2})\nu_{1,\xi\ \xi}(\frac{\xi}{2},\frac{\tau}{2}) \\ & - \frac{1}{8}\nu_{2,\xi}(\xi,\tau) - \nu_2(\xi,\tau), \end{aligned} $$

$$ \vdots $$

The rest parts of the Homotopy perturbation solution for Eq. (5.16) are thus derived by solving the aforementioned equations as follows:

$$ \nu_0(\xi,\tau) = \xi^2, $$

$$ \nu_1(\xi,\tau) = a\ \xi^2 \tau^{\delta(\xi,\tau)}, $$

$$ \nu_2(\xi,\tau) = 2\ a\ b\ \xi\ \tau^{2\delta(\xi,\tau)} - \frac{1}{4}\ b\ \xi\ \tau^{2\delta(\xi,\tau)} + c\ \xi\ \tau^{2\delta(\xi,\tau)}, $$

$$ \begin{aligned} \nu_3(\xi,\tau) =& \frac{\Gamma(2\delta(\xi,\tau)+1)}{\Gamma(3\delta(\xi,\tau)+1)}\tau^{2\delta(\xi,\tau)}[(c\ 2^{-}2\delta(\xi,\tau)\xi + 2\ a\ b2^{-1-2\delta(\xi,\tau)} - \frac{1}{4}\ b\ 2^{-1-2\delta(\xi,\tau)} + a^2 2^{-2\delta(\xi,\tau)}\xi \\ & - \frac{1}{4}ab + \frac{1}{16}b - \frac{1}{4}c\xi - 2ab\xi + \frac{1}{4}b\xi - c\xi^2)], \end{aligned} $$

$$ \vdots $$

where

$$ a = \frac{-1}{\Gamma(\delta(\xi,\tau)+1)}, \qquad b = \frac{\Gamma(\delta(\xi,\tau)+1)}{\Gamma(2\delta(\xi,\tau)+1)} \qquad c = \frac{1}{2\Gamma(\delta(\xi,\tau)+1)} \tag{4.34} $$

The residuum parts of the Homotopy perturbation solution can be produced in the same way for the subsequent components. The approximate mth-order answer to Equation (4.32).

$$\nu(\xi,\tau) \simeq \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + \nu_3(\xi,\tau) + ... \quad (4.35)$$

The residuum parts of the Homotopy perturbation solution can be produced in the same way for the subsequent components. The approximate n^{th}-order solution to Equation (4.32).

$$\nu(\xi,\tau) \simeq \nu_0(\xi,\tau) + \nu_1(\xi,\tau) + \nu_2(\xi,\tau) + \nu_3(\xi,\tau) + ... + \nu_n(\xi,\tau). \quad (4.36)$$

Figures 4.9 - 4.12 represent the approximate solution and the absolute error of Example 4.3 for $N = 3$ and different values of $\delta(\xi,\tau)$ at $\xi = 0.7$ respectively. Finally the absolute error for different values of (ξ,τ) is given in Table 4.3.

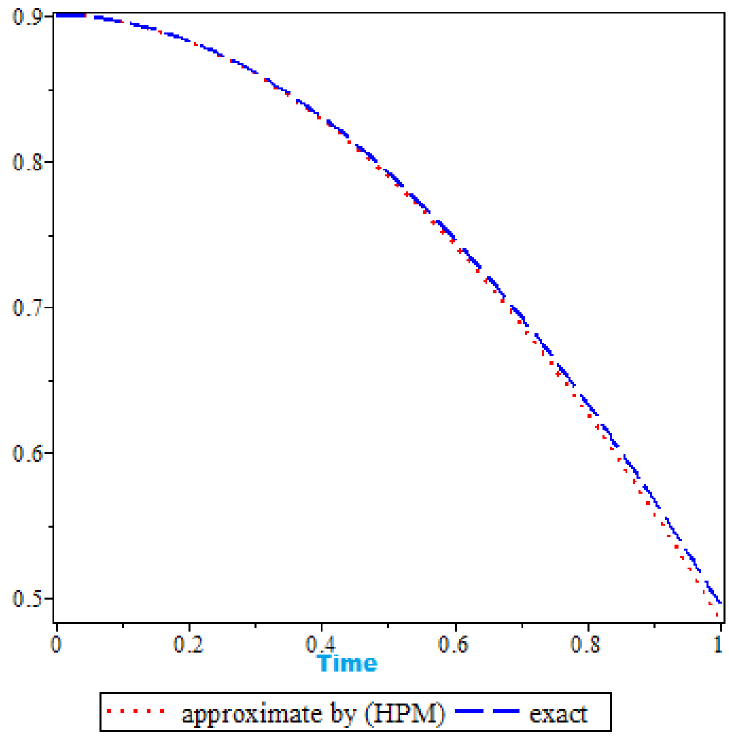

Figure 4.9: Solution of example 4.3 when $\delta(\xi,\tau) = 2 - 0.2sin(\xi\ \tau)$

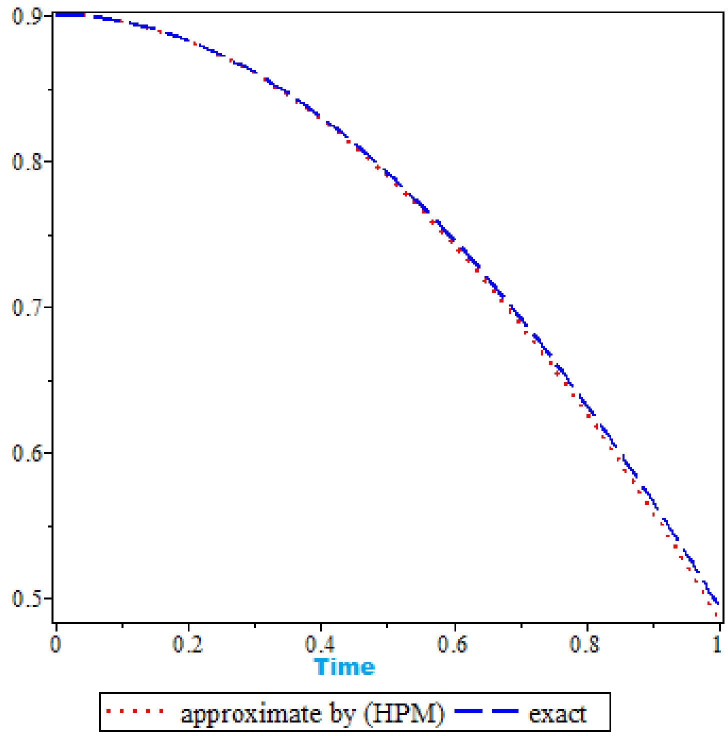

Figure 4.10: Solution of example 4.3 when $\delta(\xi,\tau) = 1.7 + 0.3e^{(-\xi\ \tau)}$

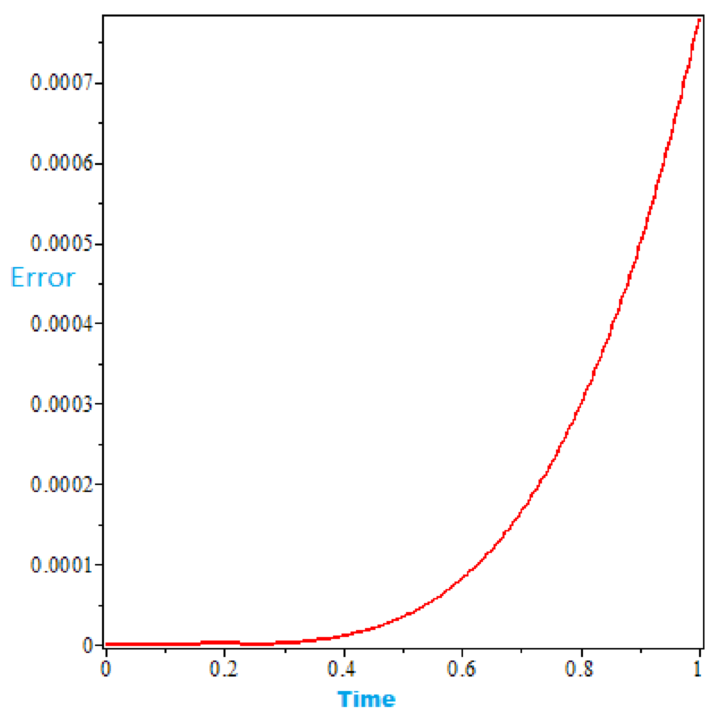

Figure 4.11: Absolute error of example 4.3 at $\xi = 0.8$ when $\delta(\xi,\tau) = 2 - 0.2sin(\xi\ \tau)$

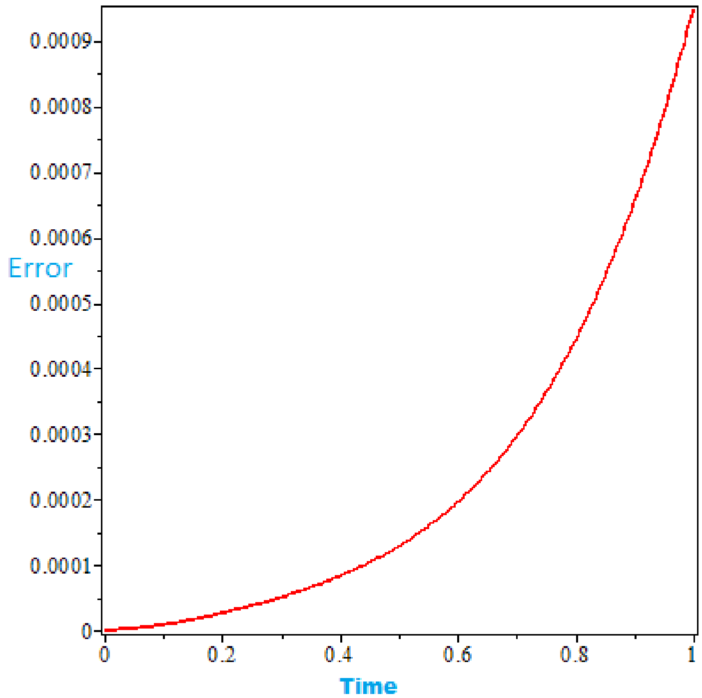

Figure 4.12: Absolute error of example 4.3 at $\xi = 0.8$ when $\delta(\xi, \tau) = 2 - 0.2sin(\xi\ \tau)$

Table 4.3: Absolute error for example 4.3 at $\delta(\xi, \tau) = 2 - 0.3e^{(\xi\ \tau)}$

(ξ, τ)	$AbsoluteError$
$(0.25, 0.25)$	1.817×10^{-4}
$(0.25, 0.50)$	5.291×10^{-4}
$(0.25, 0.75)$	1.649×10^{-4}
$(0.5, 0.25)$	3.871×10^{-3}
$(0.50, 0.50)$	3.986×10^{-4}
$(0.50, 0.75)$	2.949×10^{-4}
$(0.75, 0.25)$	1.220×10^{-4}
$(0.75, 0.50)$	2.670×10^{-3}
$(0.75, 0.75)$	4.087×10^{-4}

Chapter 5

Shifted Chebyshev operational matrices to solve the fractional time-delay diffusion equation

Introduction

This chapter consists of five sections. In section one we describe the shifted Chebyshev polynomials, while section two is related to variable-order operational matrix. Our approach will be presented in section three. Convergence analysis is given in section four. Finally, numerical examples are considered in section five.

5.1 The shifted Chebyshev polynomials

Chebyshev polynomials are defined in $[-1,1]$ and may be generated from the three terms recurrence relation

$$\mathsf{T}_0(z) = 1, \mathsf{T}_1(z) = z$$

$$\mathsf{T}_{j+1}(z) = 2z\,\mathsf{T}_j\,(z) - \mathsf{T}_{j-1}(z), \qquad j \geq 2.$$

Now, ponder a transformation $z = \{\frac{2\xi}{l} - 1\}$ to get Chebyshev's polynomials conversion in $\xi \in [0,l]$ and they are called shifted Chebyshev polynomials. The symbols for these orthogonal polynomials are $\mathsf{T}^l_j(\xi)$. Through the use of compensation for the value of z in previous relations

$$\mathsf{T}^l_0(\xi) = 1, \mathsf{T}^l_1(\xi) = \frac{2\xi}{l} - 1$$

$$\mathsf{T}^l_{j+1}(\xi) = (\frac{4\xi}{l} - 2)\,\mathsf{T}^l_j\,(\xi) - \mathsf{T}^l_{j-1}(\xi), \qquad j \geq 2.$$

The orthogonality property of the shifted Chebyshev polynomials is

$$\int_0^l \mathsf{T}_i^l(\xi)\,\mathsf{T}_j^l\,(\xi)\gamma_l(\xi)d\xi = \varsigma_j \tag{5.1}$$

where

$$\gamma_l(\xi) = \frac{1}{\sqrt{l\xi - \xi^2}} \qquad and \qquad \varsigma_j = \frac{\xi_{ij}\epsilon_j\pi}{2},$$

with $\epsilon_0 = 2,\quad \epsilon_j = 1,\quad j \geq 1$. The analytical version of $\mathsf{T}_j^l(\xi)$ is provided by degree j

$$\mathsf{T}_j^l\,(\xi) = \sum_{k=0}^{j} \mathsf{T}_{j,k}^l \xi^k \tag{5.2}$$

where

$$\mathsf{T}_{j,k}^l = (-1)^{j-k}\frac{j(j+k-1)!2^{2k}}{(j-k)!(2k)!l^k}. \tag{5.3}$$

Also, we need to defined the matrix:

$$\Theta_{l,M}(\xi) = \mathsf{T}_l\;\chi_M(\xi) \tag{5.4}$$

where the matrix element of T_l are $\mathsf{T}_{j,k}^l$, and

$$\Theta_{l,M}(\xi) = [\mathsf{T}_0^l(\xi), \mathsf{T}_1^l(\xi), ...\ \mathsf{T}_M^l\,(\xi)]^T \tag{5.5}$$

$$\chi_M(\xi) = [1, \xi, \xi^2, ..., \xi^M]^T$$

Due to equation (5.1), the vector $\chi_M(\xi)$ can be stated with $\Theta_{l,M}(\xi)$ as

$$\chi_M(\xi) = \mathsf{T}_l^{-1}\Theta_{l,M}(\xi) \tag{5.6}$$

The next two relations are at the endpoints.

$$\mathsf{T}_j^l\,(0) = (-1)^j, \quad \mathsf{T}_j^l(l) = 1, \tag{5.7}$$

will be usful in the sequel. Assume $w(\xi)$ is an integrable function that can be squared with regard to the Chebyshev weight function $\gamma_l(x) \in [0, l]$. Consequently, it can be said using $\mathsf{T}_j^l(\xi)$ as

$$w(\xi) = \sum_{j=0}^{\infty} c_j\;\mathsf{T}_j^l\,(\xi) \tag{5.8}$$

The coefficients c_j are obtained from

$$c_j = \frac{1}{\varsigma_j}\int_0^l \gamma_l(\xi)w(\xi)\;\mathsf{T}_j^l\,d\xi, \qquad j = 0, 1, ... \tag{5.9}$$

The first $(M+1)$ terms can be used to approximate the function $w(\xi)$ as

$$w_M(\xi) = \sum_{j=0}^{M} c_j \, \mathsf{T}_j^l \, (\xi) = C^T \Theta_{l,M}(\xi), \tag{5.10}$$

in which the vector C gets its value from $C^T = [c_0, c_1, ...c_M]$.

5.2 Variable-order operational matrix

In this part of this chapter, we offer the variable-order shifted Chebyshev polynomials differentiation matrix in the Caputo sense. The first-order derivative of $\Theta_{l,M}(\xi)$ can be stated as follows:

$$\frac{d}{d\xi}\Theta_{l,M}(\xi) = D_l^{(1)}\Theta_{l,M}(\xi) \tag{5.11}$$

where $D_l^{(1)}$ is the functional matrix for fractional derivative and by substituting (5.4) into (5.11) we get:

$$\frac{d}{d\xi}\Theta_{l,M}(\xi) = \mathsf{T}_l \frac{d}{d\xi}\chi_M(\xi) = \mathsf{T}_l \lambda_M \chi_M(\xi) \tag{5.12}$$

where the square matrix's dimensions λ_M is $(M+1) \times (M+1)$, and this matrix is a calculation of the operational matrix of $\chi_M(\xi)$ and

$$\lambda_M = \lambda_{ij} = \begin{cases} i+j \quad for \quad i = j+1, \quad j = 0, 1, ...M \\ \\ 0, \qquad\qquad\qquad\qquad\qquad otherwise. \end{cases} \tag{5.13}$$

Employing expressions (5.12)and (5.6), one has

$$\begin{aligned} \frac{d}{d\xi}\Theta_{l,M}(\xi) &= \mathsf{T}_l \frac{d}{d\xi}\chi_M(\xi) = \mathsf{T}_l \lambda_M \, \mathsf{T}_l^{-1} \, \Theta_{l,M}(\xi) \\ &= D_l^{(1)}\Theta_{l,M}(\xi). \end{aligned} \tag{5.14}$$

Also, we can use the Eq.(5.14) to write

$$\frac{d^q}{d\xi^q}\Theta_{l,M}(\xi) = (D_l^{(1)})^q \Theta_{l,M}(\xi) = D_l^{(q)}\Theta_{l,M}(\xi), \quad q = 1, 2, ..., \tag{5.15}$$

where $q \in N$.

Next, We broaden the applicability of the operational matrix of the variable order derivative of the Chebyshev polynomials . The variable order Caputo fractional derivative of the Chebyshev vector $\Theta_{\rho,N}(\tau)$ is:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \Theta_{\rho,N}(\tau) = D_\rho^{\delta(\xi,\tau)} \Theta_{\rho,N}(\tau), \tag{5.16}$$

where $n-1 < \delta_{min} < \delta(\xi,\tau) < \delta_{max} < n$ and $D_\rho^{\delta(\xi,\tau)}$ is an $(N+1)\times(N+1)$ matrix.
Also, it can be expressed as

$$ D_\rho^{\delta(\xi,\tau)} = \mathsf{T}_\rho \beta \mathsf{T}_\rho^{-1}, \tag{5.17}$$

where T_ρ is given in Eq.(5.4) and the elements of β are $b_{i,j},\ 0 \le i,j \le N$ can be obtained from

$$ b_{i,j} = \begin{cases} \dfrac{\tau^{-\delta(\xi,\tau)}\Gamma(i+1)}{\Gamma(i+1-\delta(\xi,\tau))} & for \quad i=j, \quad j=n,n+1,...N \\ \\ 0, & otherwise. \end{cases} \tag{5.18}$$

$$ \beta = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & \ddots & 0 & ... & 0 \\ 0 & 0 & \frac{\tau^{-\delta(\xi,\tau)}\Gamma(i+1)}{\Gamma(i+1-\delta(\xi,\tau))} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \frac{\tau^{-\delta(\xi,\tau)}\Gamma(N+1)}{\Gamma(N+1-\delta(\xi,\tau))} \end{bmatrix} \tag{5.19}$$

5.3 The Approach

This section's primary objective is to find the numerical solution of the following VFPDDEs:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = F(\xi,\tau,\nu(p_0\xi,q_0\tau), D_\xi\nu(p_1\xi,q_1\tau),..., D_\xi^{(n)}\nu(p_n\xi,q_n\tau)), \overline{n} = 0,1,2,... \tag{5.20}$$

$\nu^k(\xi,0) = \eta_k(\xi)$ for $k = 0,1,2,...,n$, $\quad n < \delta(\xi,\tau) \le n+1 \quad (\xi,\tau) \in [0,1]\times[0,1]$

$\eta_k(\xi)$ is defined as an initial function $p_i, q_j \in (0,1)$ for $i,j \in N$, with initial condition

$$ \nu(\xi,0) = g_0(\xi), \qquad 0 < \xi < 1, \tag{5.21}$$

and boundary conditions

$$ \nu(0,\tau) = g_1(\tau), \qquad \nu(1,\tau) = g_2(\tau), \qquad 0 < \tau \le \rho. \tag{5.22}$$

At the beginning we will approximated $u(\xi,\tau)$ by means of the double-shifted Chebyshev polynomials as

$$\nu_{N,M}(\xi,\tau)=\sum_{i=0}^{M}\sum_{j=0}^{N}a_{i,j}\ \mathsf{T}_i^{l}\ (\xi)\ \mathsf{T}_j^{\rho}\ (\tau)=\Theta_{\rho,N}^{T}(\tau)A\Theta_{l,M}(\xi), \tag{5.23}$$

where A is the enigmatized matrix.
The use of Eqs.(5.15), (5.16) and (5.23), enable us to write

$$\begin{aligned}{}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau)&=\Theta_{\rho,N}^{T}(\tau)(D_{\rho}^{\delta(\xi,\tau)})^{T}A\Theta_{l,M}(\xi),\\ \frac{\partial^2\nu(\xi,\tau)}{\partial\xi^2}&=\Theta_{\rho,N}^{T}(\tau)(D_{l}^{2})^{T}A\Theta_{l,M}(\xi),\\ \nu(\xi,0)&=\Theta_{\rho,N}^{T}(0)A\Theta_{l,M}(\xi),\\ \nu(0,\tau)&=\Theta_{\rho,N}^{T}(\tau)A\Theta_{l,M}(0),\\ \nu(1,\tau)&=\Theta_{\rho,N}^{T}(0)A\Theta_{l,M}(1).\end{aligned} \tag{5.24}$$

replacing Eqs.(5.23) and (5.24) into Eqs.(5.20), (5.21) and (5.22) to get

$$\begin{aligned}&\Theta_{\rho,N}^{T}(\tau)(D_{\rho}^{\delta(\xi,\tau)})^{T}A\Theta_{l,M}(\xi)=\\ &F(\Theta_{\rho,N}^{T}(q_0\tau)A\Theta_{l,M}(p_0\xi),\Theta_{\rho,N}^{T}(q_0\tau)D_l^{1}A\Theta_{l,M}(p_0\xi),\Theta_{\rho,N}^{T}(q_1\tau)D_l^{2}A\Theta_{l,M}(p_1\xi)\\ &...,\Theta_{\rho,N}^{T}(q_nt)(D_l^{m})A\Theta_{l,M}(p_m\xi)),\\ &\Theta_{\rho,N}^{T}(0)A\Theta_{l,M}(\xi)=g_0(\xi),\\ &\Theta_{\rho,N}^{T}(\tau)A\Theta_{l,M}(0)=g_1(\tau),\\ &\Theta_{\rho,N}^{T}(\tau)A\Theta_{l,M}(1)=g_2(\tau).\end{aligned} \tag{5.25}$$

Suppose $\xi_i(0\leq i\leq M)$ is the nodes of the shifted Chebyshev-Gauss-Lobatto quadrature and $\tau_i(0\leq j\leq N-1)$ is zeros of $\mathsf{T}_N^{\rho}(\tau)$. This produces $\mathsf{T}_N^{\rho}(\tau)=0$, use these nodes in Eq.(5.25). Then, using the collocation method of Eq. (5.20) is

$$\begin{aligned}&\Theta_{\rho,N}^{T}(\tau_i)(D_{\rho}^{\delta(\xi_i,\tau_j)})^{T}A\Theta_{l,M}(\xi_i)=\\ &F(\Theta_{\rho,N}^{T}(q_0\tau_j)A\Theta_{l,M}(p_0\xi_i),\Theta_{\rho,N}^{T}(q_0\tau_j)D_l^{1}A\Theta_{l,M}(p_0\xi_i),\Theta_{\rho,N}^{T}(q_1\tau_j)D_l^{2}A\Theta_{l,M}(p_1\xi_i)\\ &...,\Theta_{\rho,N}^{T}(q_n\tau_j)(D_l^{m})A\Theta_{l,M}(p_m\xi_i)),\quad 0\leq i\leq M-1,\quad 0\leq j\leq N-1,\\ &\Theta_{\rho,N}^{T}(0)A\Theta_{l,M}(\xi_i)=g_0(\xi_i),\qquad 0\leq i\leq M,\\ &\Theta_{\rho,N}^{T}(\tau_j)A\Theta_{l,M}(0)=g_1(\tau_j),\qquad 0\leq j\leq N-1,\\ &\Theta_{\rho,N}^{T}(\tau_j)A\Theta_{l,M}(1)=g_2(\tau_j),\qquad 0\leq j\leq N-1.\end{aligned} \tag{5.26}$$

A linear algebraic system can be created from the aforementioned equations. We will calculating coefficients $a_{i,j}$ as a result, it is possible to assess the approximate solution of $\nu_{N,M}(\xi,\tau)$ in Eq.(5.23).

5.4 Error Analysis and Convergence

We demonstrate uniform convergence of the Shifted Chebyshev expansion of the continuous function $\nu(\xi,\tau)$. But first, using the following theorem, we provide the upper bound on its error. Let $P_{M,N}$ be the set of all polynomials with degree no greater than M for the variable ξ and no greater than N for the variable ρ. Thus, for $\nu \in C(\Omega)$, there exists unique $p_{M,N} \in P_{M,N}$ such that

$$||\nu(\xi,\tau) - \nu_{M,N}(\xi,\tau)||_{L^2(\Omega)} \leq ||\nu(\xi,\tau) - P_{M,N}(\xi,\tau)||_{L^2(\Omega)}. \tag{5.27}$$

Since

$$L^2(\Omega) = \{\vartheta : \vartheta \ is \ measurableon \ \Omega \ and \ ||\vartheta||_\omega < \infty\},$$

equipped and the following inner product and norm define as:

$$\langle\vartheta,\rho\rangle = \int_\Omega \vartheta(\xi,\tau)\rho(\xi,\tau)\omega(\xi,\tau)d\xi d\tau, \qquad ||\vartheta||_\omega = \langle\vartheta,\vartheta\rangle_\omega.$$

Theorem 5.4.1:
Assume that the Shifted Chebyshev functions expand the real sufficiently smooth function ν in Ω , as

$$\nu_{M,N}(\xi,\tau) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N} a_{i,j} \ \mathsf{T}_i^l\,(\xi)\ \mathsf{T}_j^\rho\,(\tau) = \Theta_{\rho,N}^T(\tau)A\Theta_{l,M}(\xi),$$

where $\Theta_{\rho,N}^T(\tau)$, $\Theta_{l,M}(\xi)$ are defined in (5.5)
if the rough answer achieved using the technique suggested in Section 4, then we have

$$||\nu(\xi,\tau) - \nu_{M,N}(\xi,\tau)||_{L^2(\Omega)} \leq K_{M\ N}\sqrt{\frac{(2N+2)!}{(2M+3)}} + \mathcal{C}_{M\ N}\|\nu - \hat{\nu}\|_{L^2}. \tag{5.28}$$

proof: see[78]

5.5 Illustrative Examples

In this section, the numerical approach given in section 5.3 will be performed for several VFPDDEs.

Example 5.1.

Consider this proportional delay modified time-fractional Burgers equation's:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \nu(\xi, \frac{\tau}{3})\ \nu(\xi, \frac{\tau}{3}) - \frac{1}{3} \nu(\xi,\tau), \tag{5.29} $$

where $\xi, \tau \in [0,1]$ and $\delta(\xi,\tau) = 2 - 0.3sin(\xi\ \tau)$ with initial conditions $\nu(x,0) = \xi^2$ and $p_0 = p_2 = 1, q_0 = q_2 = \frac{1}{3}$.
This equation has exact solution that is $\nu(\xi,t) = \xi^2 cosh(1.3t)$. By using the technique given in section 5.3, the equation (5.29) becomes:

$$ \Theta_{\rho,N}^T(\tau_i)(D_\rho^{\delta(\xi_i,\tau_j)})^T A \Theta_{l,M}(\xi_i) = $$
$$ \Theta_{\rho,N}^T(\frac{t_j}{3}) D_l^2 A \Theta_{l,M}(\xi_i)\ \Theta_{\rho,N}^T(\frac{\tau_j}{3}) A \Theta_{l,M}(\xi_i) - \frac{1}{3} \Theta_{\rho,N}^T(\tau_j) A \Theta_{l,M}(\xi_i), $$

and

$$ \Theta_{\rho,N}^T(0) A \Theta_{l,M}(\xi_i) = g_0(\xi_i), $$

$$ \Theta_{\rho,N}^T(\tau_j) A \Theta_{l,M}(0) = g_1(\tau_j), $$

$$ \Theta_{\rho,N}^T(\tau_j) A \Theta_{l,M}(1) = g_2(\tau_j). $$

By solving the above equations with $\rho = 0.5, l = 10$, we obtain a system of non-liner equations, the goal of solving these equations is to find the unknown matrix A, after obtaining the approximate solution by the proposed method we display $\|e\|_{l^2}$ to set the absolute error see Table 5.1. Also, Figs. 5.1 and 5.2 are comparison between the approximate and the exact solution for different values of $\delta(\xi,\tau)$ and in Figs. 5.3 and 5.4 we plot the error functions at $N = M = 8$ with different values of $\delta(\xi,\tau)$. Finally, Figs 5.5 and 5.6 represent approximate and the exact solution.

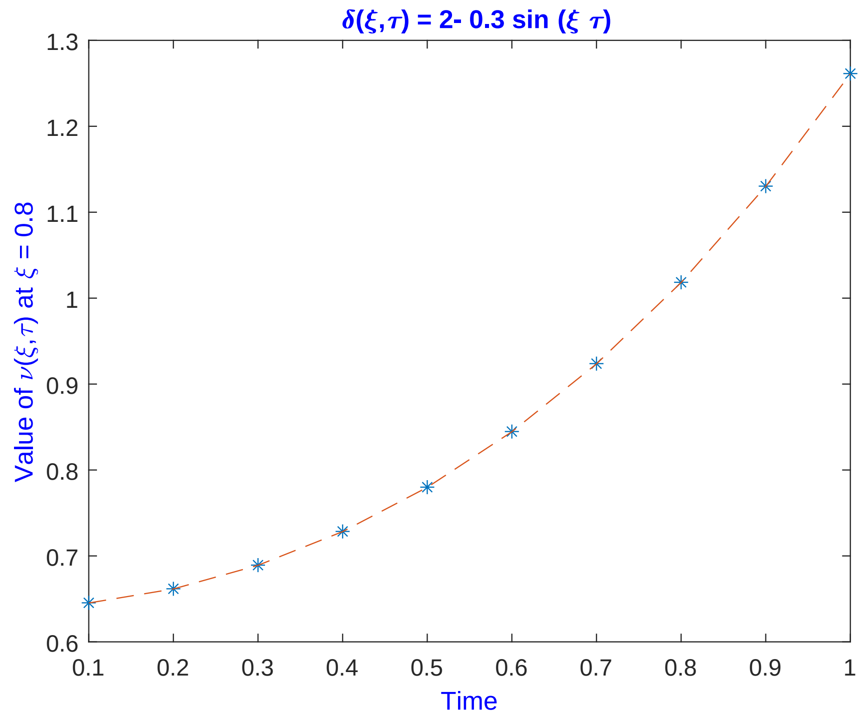

Figure 5.1: Approximate and exact solution of example 5.1.

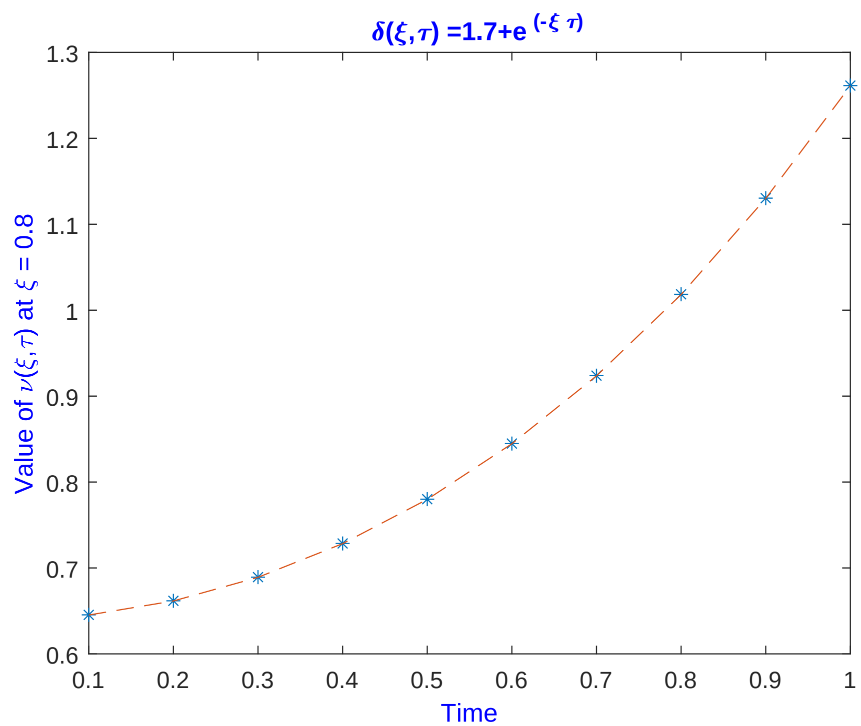

Figure 5.2: Approximate and exact solution of example 5.1.

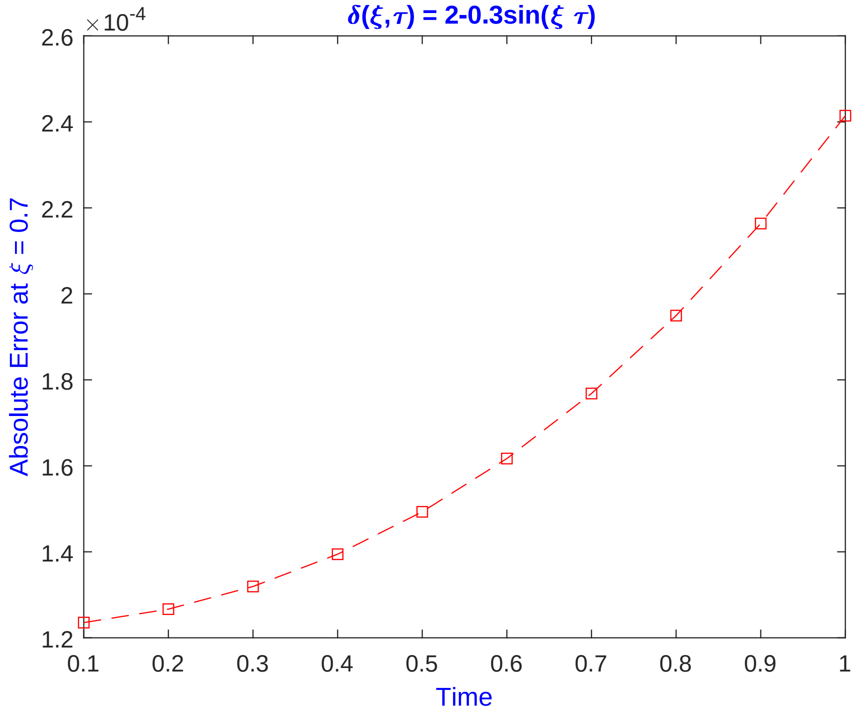

Figure 5.3: The absolute error of Example 5.1.

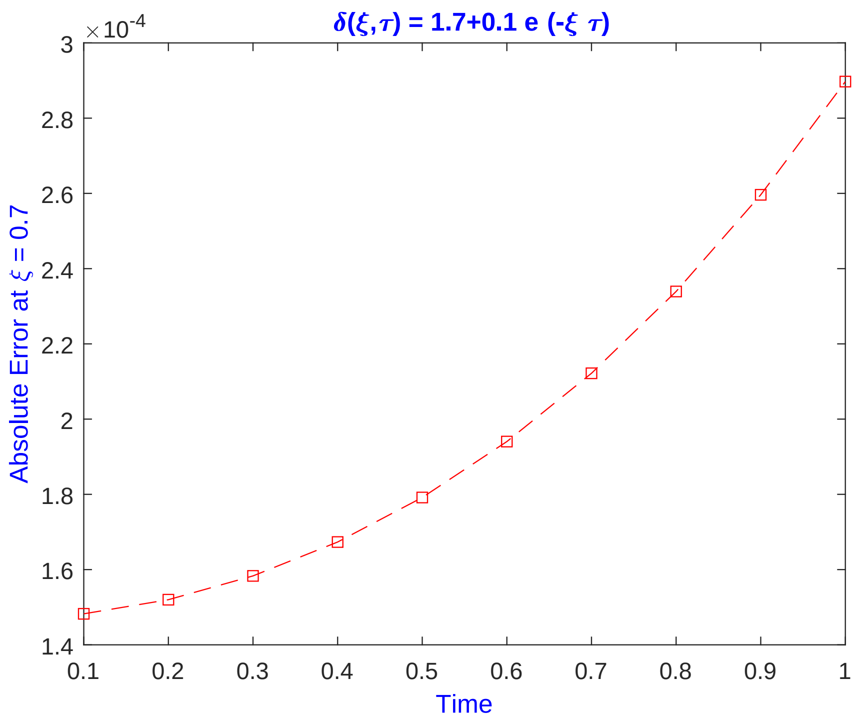

Figure 5.4: The absolute error of Example 5.1.

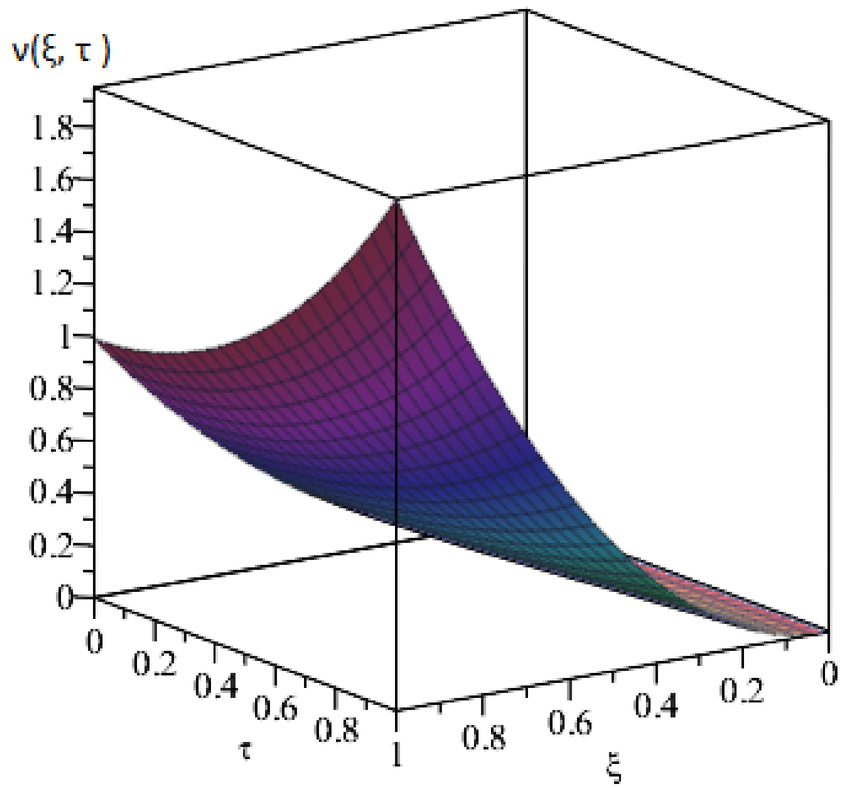

Figure 5.5: Approximate solution of Example 5.1.

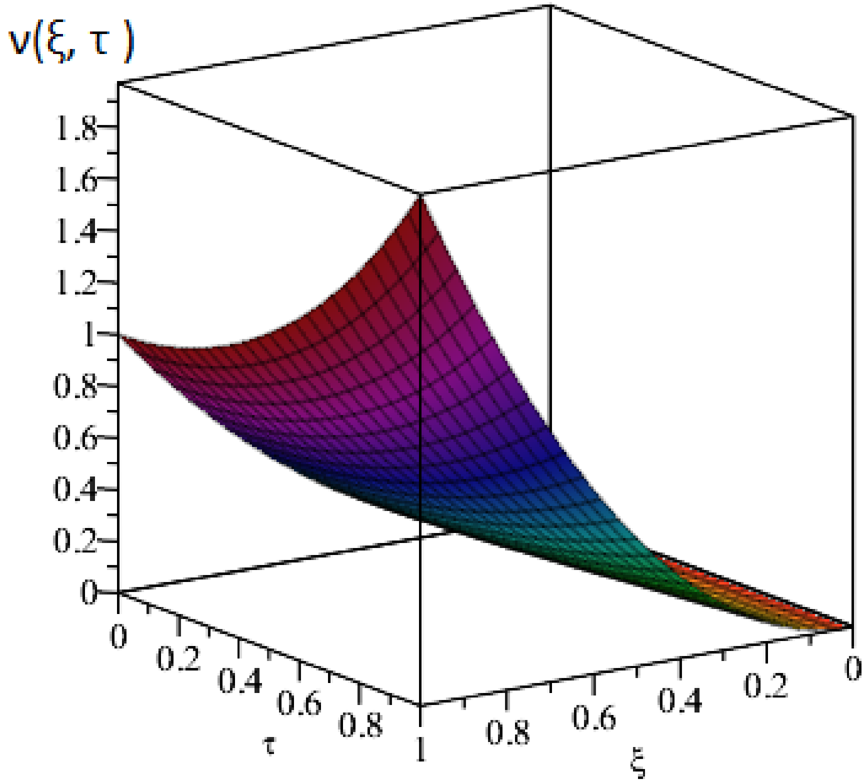

Figure 5.6: Exact solution of Example 5.1.

Table 5.1: *The absolute error of Example* 5.1 *at* $\xi = 0.8$

τ	$\delta(\xi,\tau) = 1.7 + 0.1e^{(-\xi\ \tau)}$	$\delta(\xi,\tau) = 2 - 0.3\ sin(\xi\ \tau)$
0.1	1.4845×10^{-4}	1.2908×10^{-4}
0.2	1.5220×10^{-4}	1.3235×10^{-4}
0.3	1.5854×10^{-4}	1.3786×10^{-4}
0.4	1.6755×10^{-4}	1.4570×10^{-4}
0.5	1.7941×10^{-4}	1.5601×10^{-4}
0.6	1.9430×10^{-4}	1.6895×10^{-4}
0.7	2.1247×10^{-4}	1.8476×10^{-4}
0.8	2.3424×10^{-4}	2.0369×10^{-4}
0.9	2.5998×10^{-4}	2.2697×10^{-4}

Example 5.2.

Think about the simplified time-fractional Burgers equation for proportional delay:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)}\nu(\xi,\tau) = \frac{\partial^2}{\partial\xi^2}\nu(\xi,\tau) + \nu(\xi,\frac{\tau}{2})\nu(\frac{\xi}{2},\frac{\tau}{2}) + \frac{1}{2}\nu(\xi,\tau), \quad (5.30) $$

with initial conditions $\nu(\xi,0)=\xi$, $\delta(\xi,\tau)=1.7+0.3cos^2(\xi\ \tau)$ and $p_0=q_2=1, p_0=q_0=\frac{1}{2}$. The exact solution of this problem is $\nu(\xi,\tau)=\xi e^{0.5\tau}$.

By using technique given in section 5.3, the equation (5.30) become

$$ \Theta_{\rho,N}^T(\tau_i)(D_\rho^{\delta(\xi_i,\tau_j)})^T A\Theta_{l,M}(\xi_i) = $$

$$ \Theta_{\rho,N}^T(\tau_j)D_l^2 A\Theta_{l,M}(\xi_i) + \Theta_{\rho,N}^T(\frac{\tau_j}{2})A\Theta_{l,M}(\xi_i)\Theta_{\rho,N}^T(\frac{\tau_j}{2})A\Theta_{l,M}(\frac{\xi_i}{2}) + \frac{1}{2}\Theta_{\rho,N}^T(\tau_j)A\Theta_{l,M}(\xi_i), $$

and

$$ \Theta_{\rho,N}^T(0)A\Theta_{l,M}(\xi_i) = g_0(\xi_i), $$

$$ \Theta_{\rho,N}^T(\tau_j)A\Theta_{l,M}(0) = g_1(\tau_j), $$

$$ \Theta_{\rho,N}^T(\tau_j)A\Theta_{l,M}(1) = g_2(\tau_j). $$

By solving the above equations with $\rho=0.5, l=10$, we obtain a system of non-liner equations, the goal of solving these equations is to find the unknown matrix A, after obtaining the approximate solution by the proposed method we display $\|e\|_{l^2}$ to set the absolute error see Table 5.2. Also, Figs. 5.7 and 5.8 are comparison between the approximate and the exact solution for different values of $\delta(\xi,\tau)$ and in Figs. 5.9 and 5.10 we plot the error functions at $N=M=8$ with different values of $\delta(\xi,\tau)$. Finally, Figs 5.11 and 5.12 represent approximate and the exact solution.

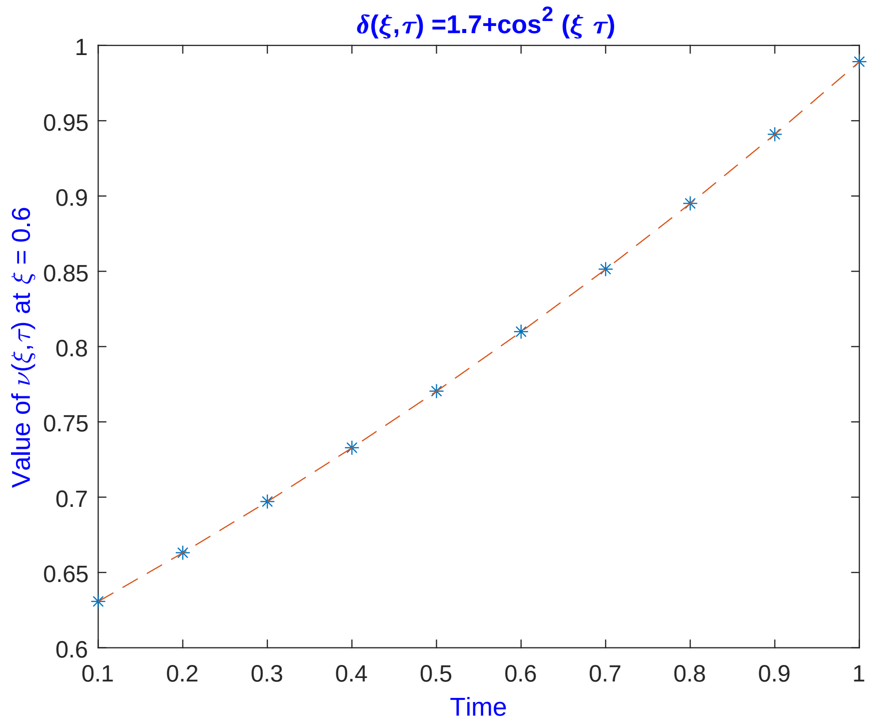

Figure 5.7: Approximate and exact solution of example 5.2.

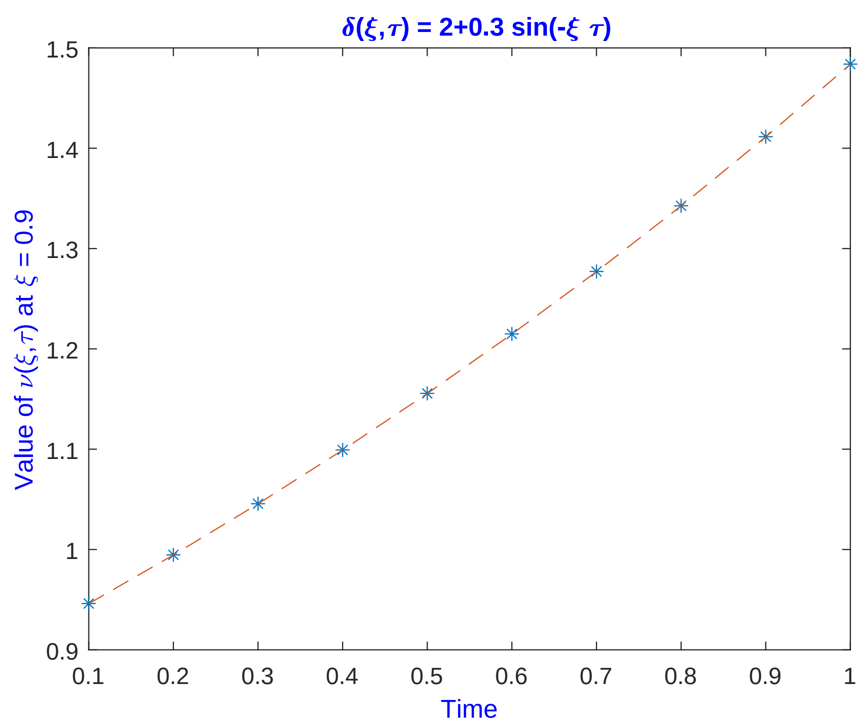

Figure 5.8: Approximate and exact solution of example 5.2.

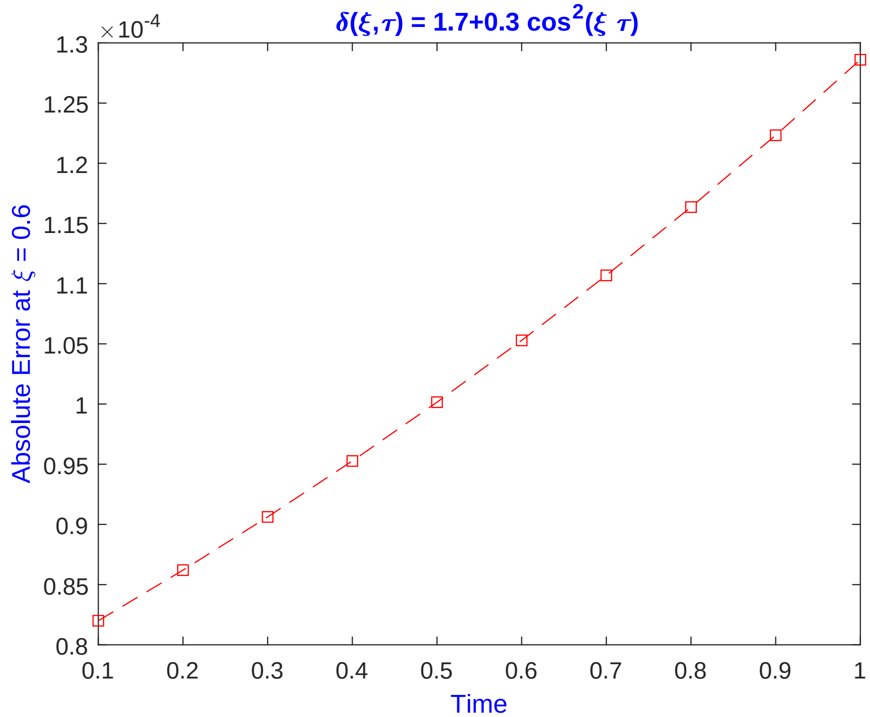

Figure 5.9: The absolute error of example 5.2.

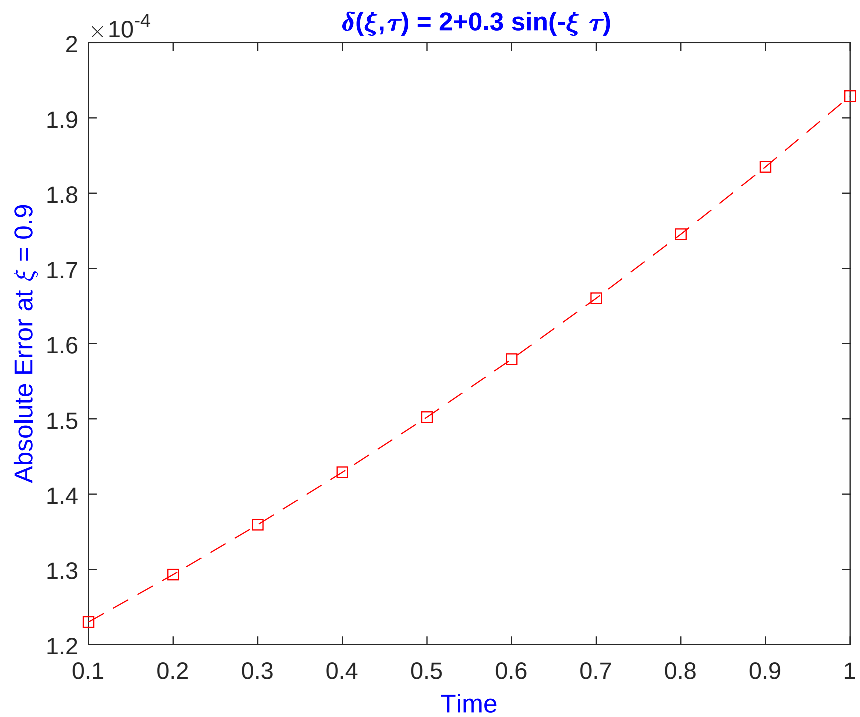

Figure 5.10: The absolute error of example 5.2.

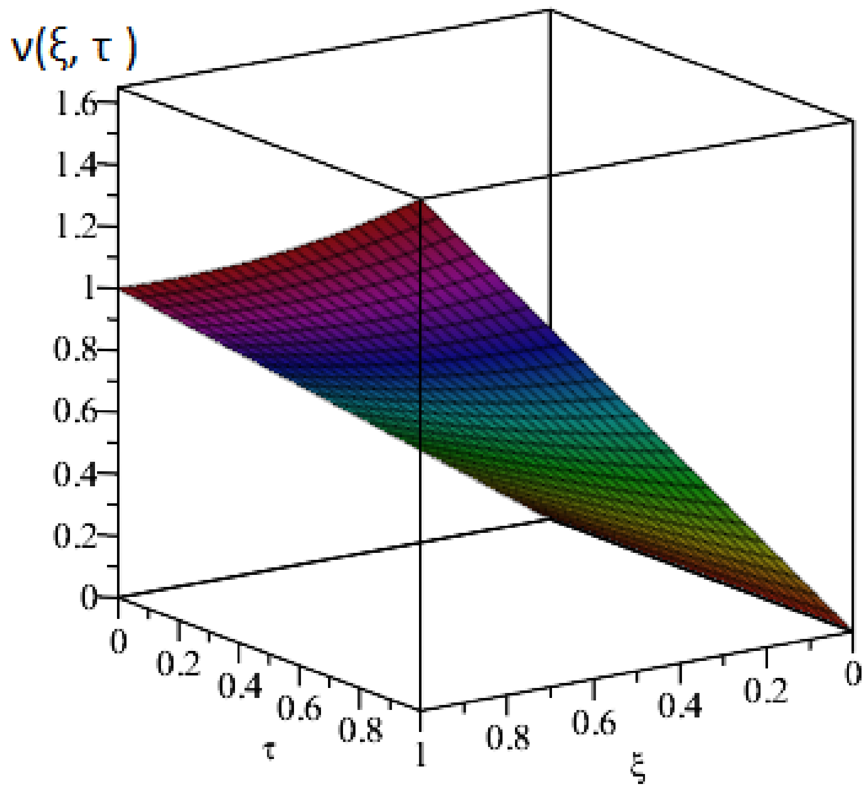

Figure 5.11: Exact solution of example 5.2.

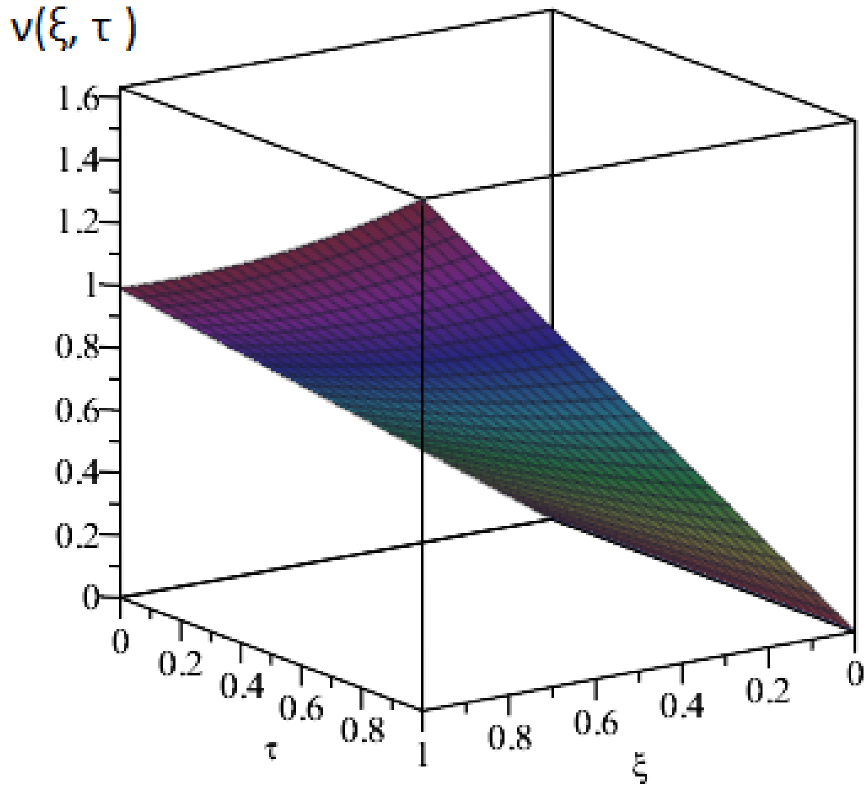

Figure 5.12: Approximate solution of example 5.2.

Table 5.2: The absolute error of example 5.2

τ	$\delta(\xi,\tau) = 1.7 + 0.3\ cos^2(\xi\ \tau)$	$\delta(\xi,\tau) = 2 + 0.2\ sin(-\xi\ \tau)$
0.1	8.1999×10^{-5}	1.2300×10^{-4}
0.2	8.6203×10^{-5}	1.2930×10^{-4}
0.3	9.0623×10^{-5}	1.3593×10^{-4}
0.4	9.5269×10^{-5}	1.4290×10^{-4}
0.5	1.0015×10^{-4}	1.5023×10^{-4}
0.6	1.0529×10^{-4}	1.5793×10^{-4}
0.7	1.1069×10^{-4}	1.6603×10^{-4}
0.8	1.1636×10^{-4}	1.7454×10^{-4}
0.9	1.2233×10^{-4}	1.8349×10^{-4}

Example 5.3.

Considering what follows proportional delay FPDE:

$$ {}_{0}^{C}D_{\tau}^{\delta(\xi,\tau)}\nu(\xi,\tau) = \frac{\partial^2}{\partial\xi^2}\nu(\frac{\xi}{2},\frac{\tau}{2})\frac{\partial}{\partial\xi}\nu(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\frac{\partial}{\partial\xi}\nu(\xi,\tau) - \nu(\xi,\tau), \quad (5.31) $$

$\xi,\tau \in [0,1]$ and $\delta(\xi,\tau) = 2 - 0.3\ exp(\xi\ \tau)$ with basic conditions $\nu(\xi,0) = \xi^2$ and $p_0 = q_0 = 1, p_1 = q_1 = p_2 = q_2 = \frac{1}{2}$.
The exact solution of this problem is $\nu(\xi,\tau) = \xi^2 cos(\tau)$.

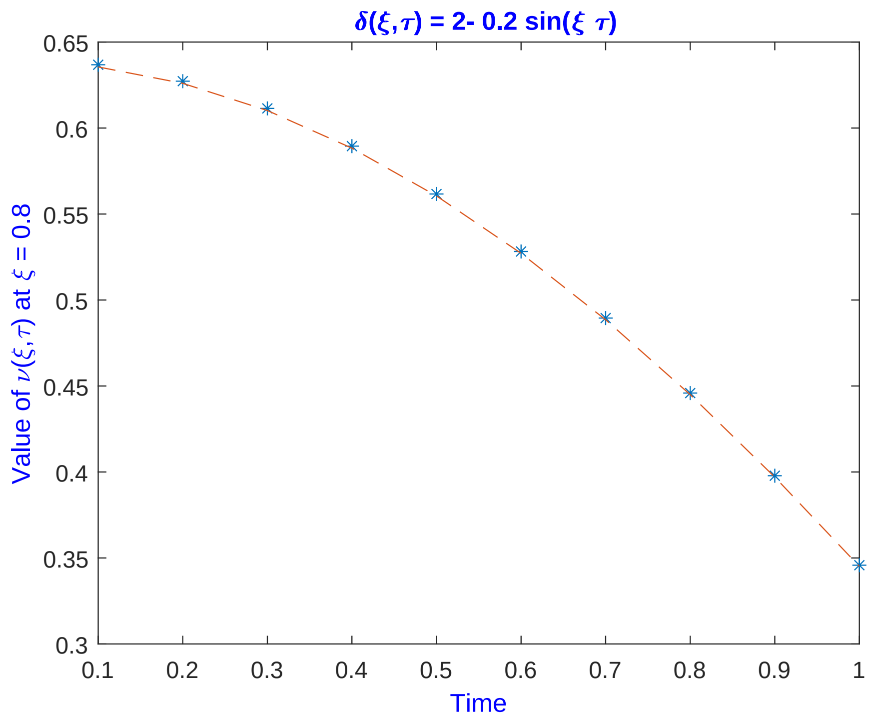

Figure 5.13: Approximate and exact solution of example 5.3.

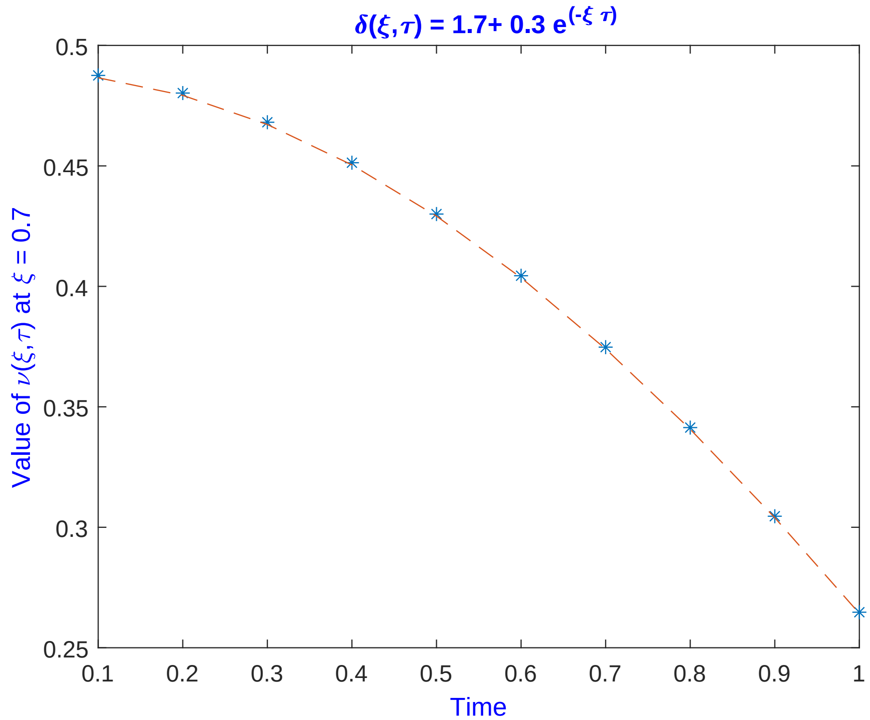

Figure 5.14: Approximate and exact solution of example 5.3.

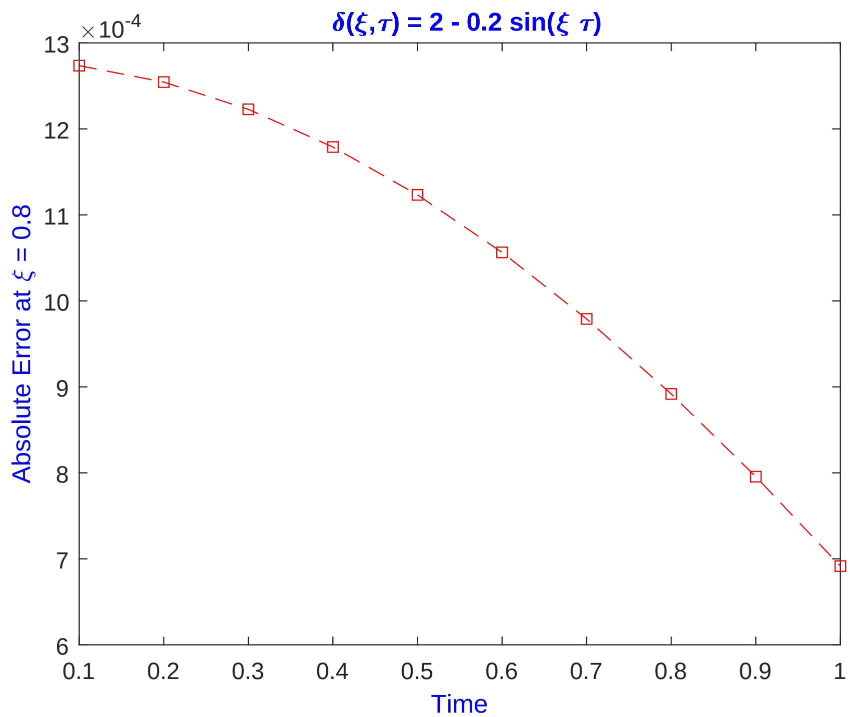

Figure 5.15: The absolute error of example 5.3.

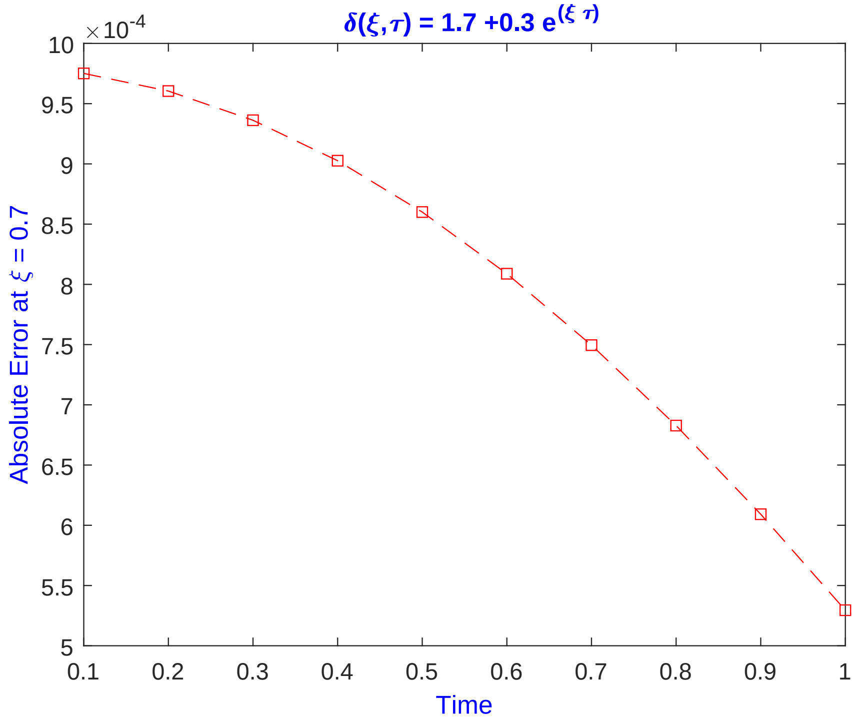

Figure 5.16: The absolute error of example 5.3.

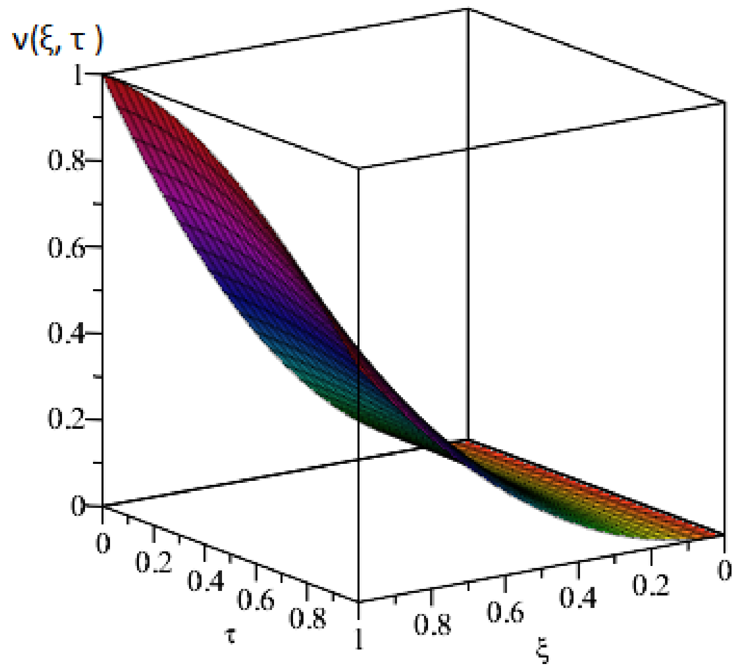

Figure 5.17: Exact solution of example 5.3.

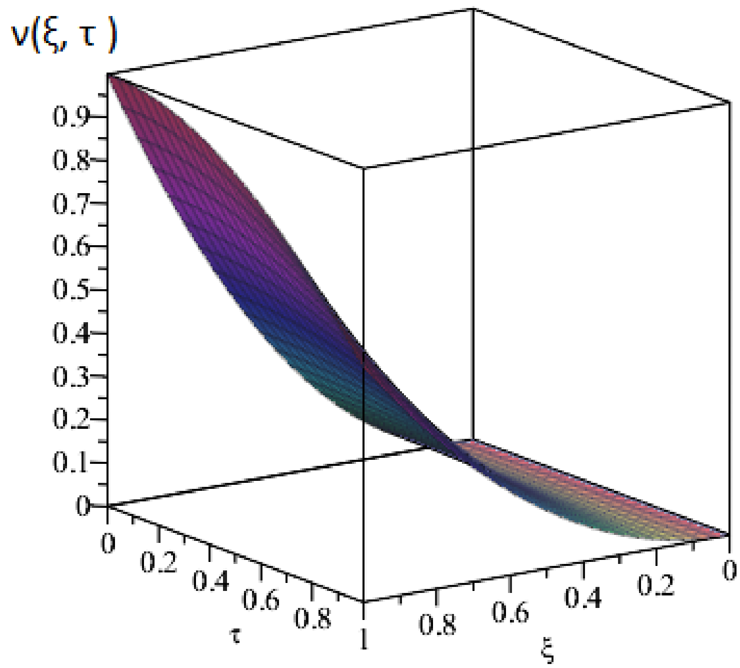

Figure 5.18: Approximate solution of example 5.3.

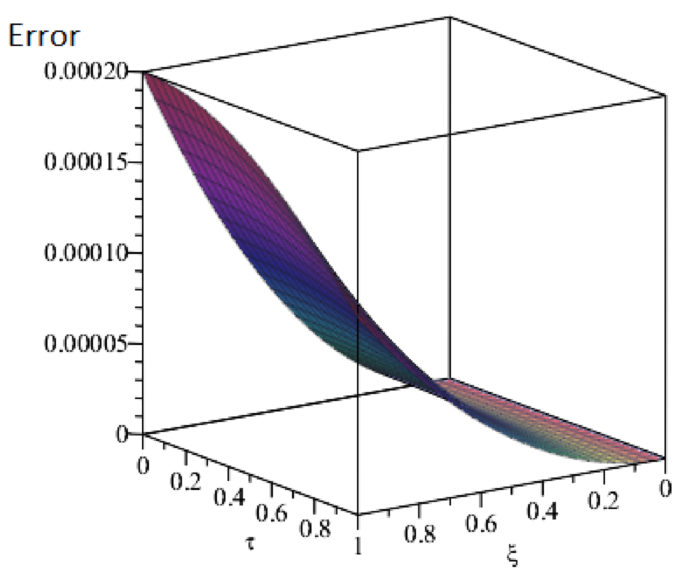

Figure 5.19: Absolute error of example 5.3 at $\delta(\xi, \tau) = 1.7 + 0.3cos^2(\xi\tau)$.

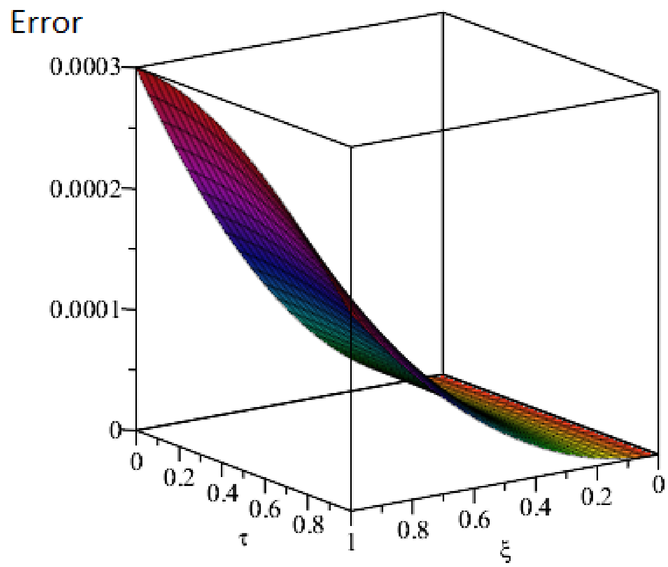

Figure 5.20: Absolute error of example 5.3 $\delta(\xi,\tau) = 2 + 0.2sin(-\xi\tau)$.

Table 5.3: The absolute error of example 5.3

τ	$\delta(\xi,\tau) = 1.7 + 0.3\ e^{(-\xi\ \tau)}$	$\delta(\xi,\tau) = 2 - 0.2\ sin(\xi\ \tau)$
0.1	1.4627×10^{-4}	1.3000×10^{-3}
0.2	1.4407×10^{-4}	1.3000×10^{-3}
0.3	1.4043×10^{-4}	1.2000×10^{-3}
0.4	1.3540×10^{-4}	1.2000×10^{-3}
0.5	1.2900×10^{-4}	1.1000×10^{-3}
0.6	1.2132×10^{-4}	1.1000×10^{-3}
0.7	1.1243×10^{-4}	9.7900×10^{-4}
0.8	1.0242×10^{-4}	8.9178×10^{-4}
0.9	9.1377×10^{-5}	7.9566×10^{-4}

Figures 5.3 - 5.16 show approximate results for the example 5.3 for $N = 8$ and more than the value of $\delta(\xi,\tau)$ at $\xi = 0.7$. At last, the standard error for many values of $\delta(\xi,\tau)$ is demonstrated in Table 5.3.

Example 5.4.

Considering what follows proportional delay FPDE:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \nu(\xi, \frac{\tau}{2}) \nu(\xi, \frac{\tau}{2}) - \nu(\xi,\tau), \tag{5.32} $$

$\xi, \tau \in [0,1]$ and $\delta(\xi,\tau) = 2 - 0.1e^{-\xi\tau}$ with basic conditions $\nu(x,0) = \xi^2$.
The exact solution of this problem is $\nu(\xi,\tau) = \xi^2 e^{\tau}$. Figures 5.21 and 5.22 show approximate results for the example 5.4 for $N = 8$ at different kinds of $\delta(\xi,\tau)$ at $\xi = 0.8$. Also, figures 5.23 and 5.24 show the absolute error of different value of variable order fractional.

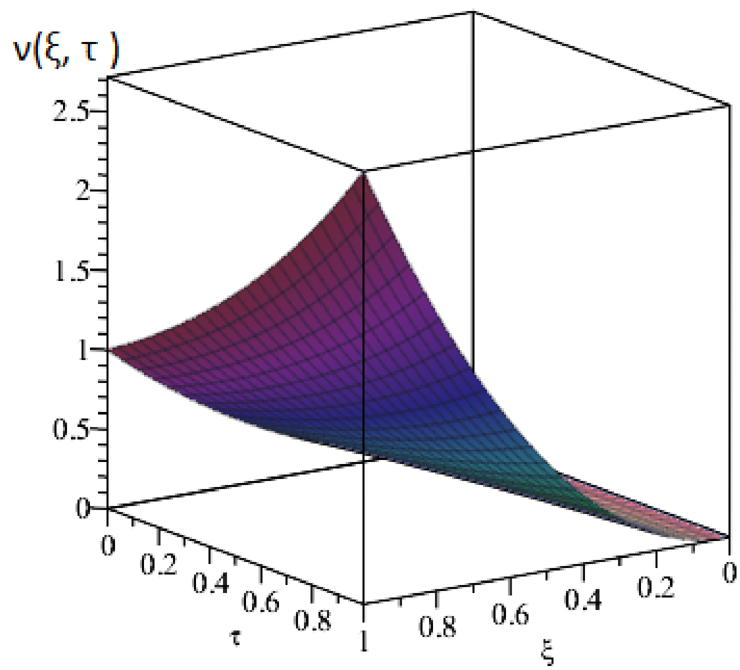

Figure 5.21: Exact solution of example 5.4.

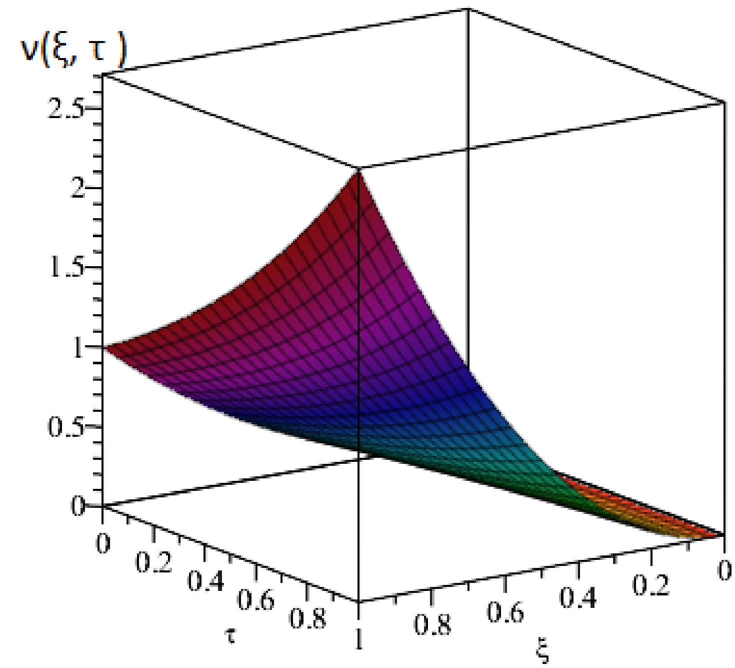

Figure 5.22: Approximate solution of example 5.4.

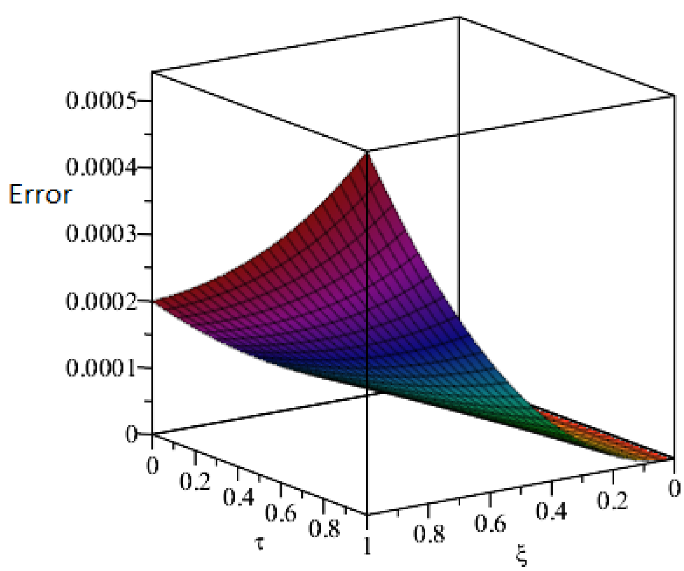

Figure 5.23: Absolute error of example 5.4 at $\delta(\xi, \tau) = 2 - 0.1e^{-\xi\tau}$.

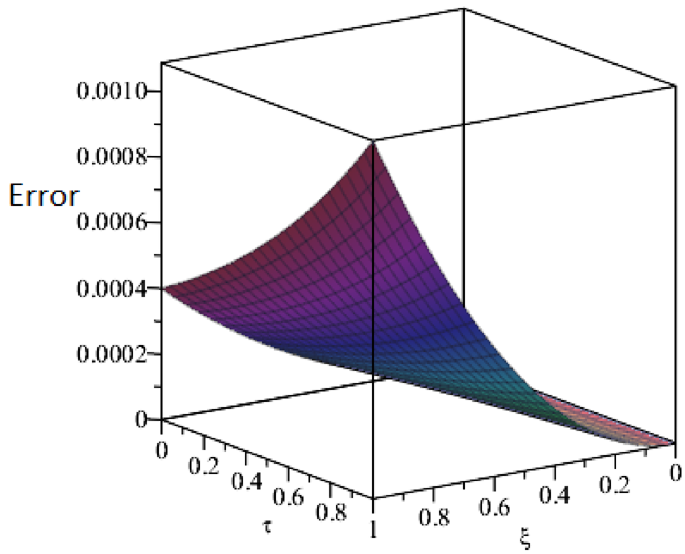

Figure 5.24: Absolute error of example 5.4 at $\delta(\xi,\tau) = 2 - 0.2sin(\xi\tau)$.

Example 5.5.

Considering what follows proportional delay FPDE:

$$ {}_0^C D_\tau^{\delta(\xi,\tau)} \nu(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \nu(\frac{\xi}{2},\frac{\tau}{2}) \frac{\partial}{\partial \xi} \nu(\frac{\xi}{2},\frac{\tau}{2}) - \frac{1}{8}\frac{\partial}{\partial \xi}\nu(\xi,\tau) - \nu(\xi,\tau), \tag{5.33} $$

$\xi,\tau \in [0,1]$ and $\delta(\xi,\tau) = 1 + 0.5e^{-\xi}sin(\tau)$ with basic conditions $\nu(x,0) = \xi^2 + 2$. The exact solution of this problem is $\nu(\xi,\tau) = (\xi^2+2)e^{-\tau}$.
Figures 5.25 and 5.26 show approximate results for the example 5.5 for $N = 8$ and more than the value of $\delta(\xi,\tau)$ at $\xi = 0.7$. Also, figures 5.27 and 5.28 show the absolute error of different value of variable order fractional.

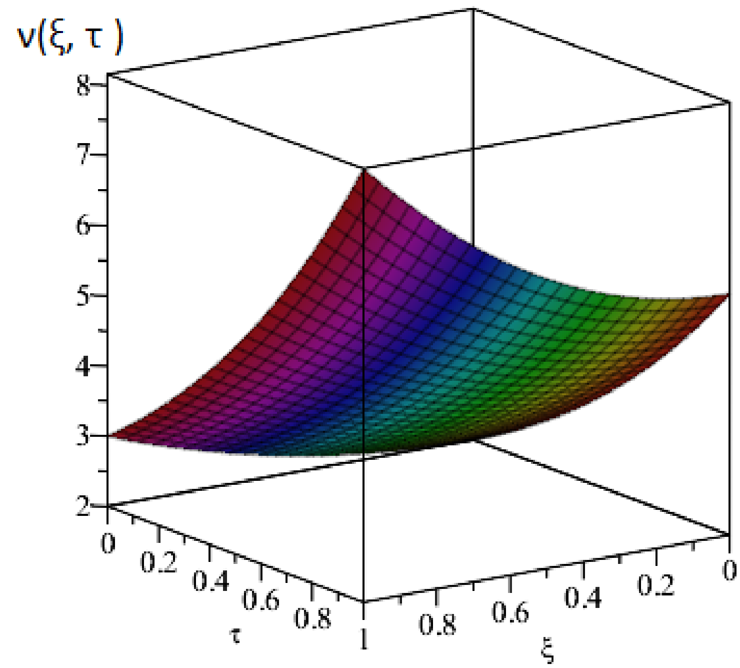

Figure 5.25: Exact solution of example 5.5.

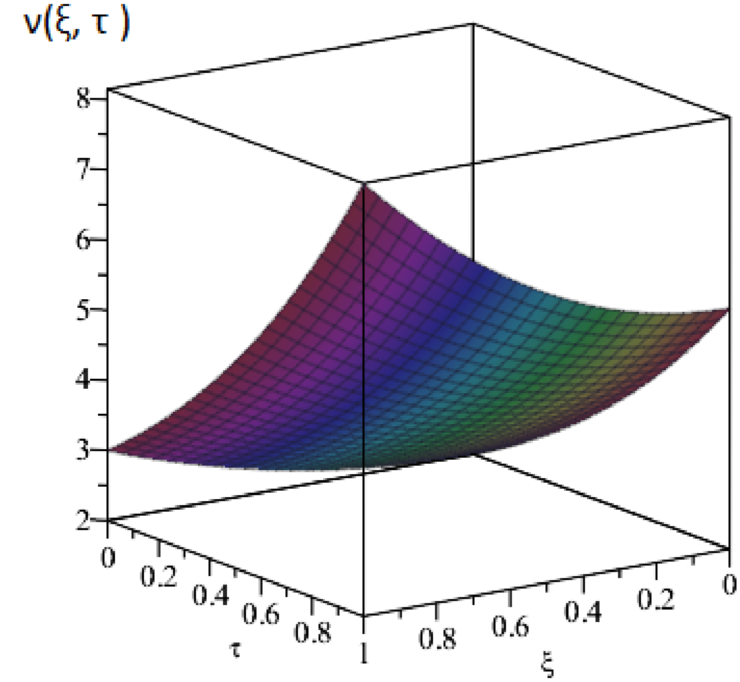

Figure 5.26: Approximate solution of example 5.5.

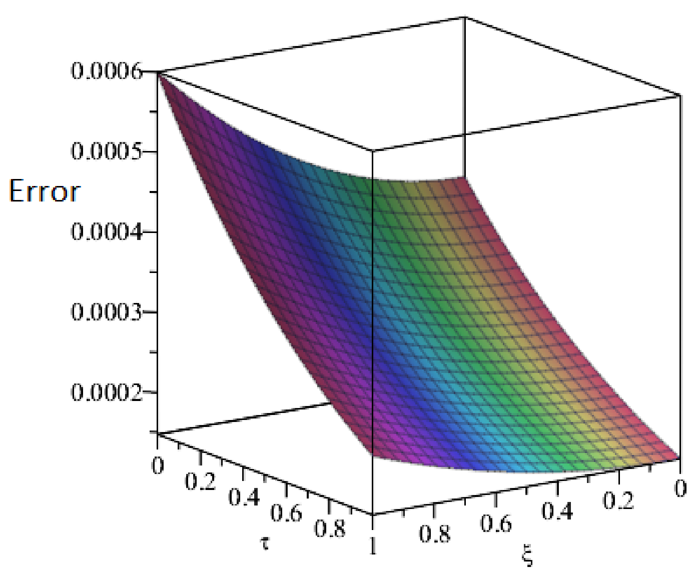

Figure 5.27: Absolute error of example 5.5 at $\delta(\xi, \tau) = 1 + 0.5e^{-\xi} sin(\tau)$.

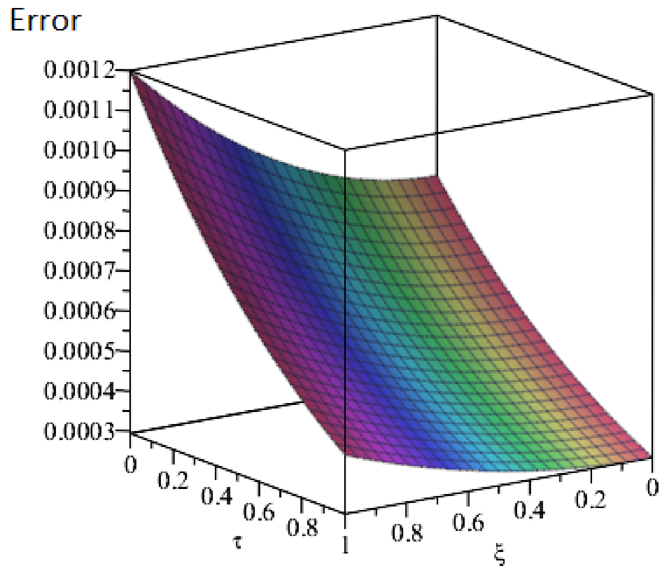

Figure 5.28: Absolute error of example 5.5 at $\delta(\xi,\tau) = 1.7 + 0.2e^{-\xi\tau}$.

Chapter 6

Solving Fractional Time-Delay Diffusion Equation with Variable-Order Derivative Based on Shifted Chebyshev-Laguerre Operational Matrices

Introduction

This chapter consists of seven sections. Section one will be related to the problem statement. In section two the function approximation is given. Section three and four are about the operational matrices of integral (integer and variable) of the shifted Chebyshev and Lagurre polynomials. Our approach will be presented in section five. The convergence analysis is given in section six. Finally numerical examples are considered in section seven.

6.1 Problem statement

In this section, we state the variable order fractional time-delay diffusion equation that will be handled and analyzed in the next sections as follows:

$$D_{\tau}^{\delta(\xi,\tau)}\ \nu(\xi,\tau) - \eta\ \frac{\partial^2\nu(\xi,\tau)}{\partial\ \xi^2} = f(\tau,\nu(\xi,\tau),\nu(\xi,\tau-\kappa)), \tag{6.1}$$

$$0 \leq \xi \leq 1, \quad 0 < \tau \leq \infty.$$

Subject to :

$$\nu(0,\tau) = \nu_0(\tau), \quad \nu(1,\tau) = \nu_1(\tau). \tag{6.2}$$

and

$$\nu(\xi,0)=g_0(\xi), \qquad \frac{\partial \nu(\xi,0)}{\partial \tau}=g_1(\xi). \tag{6.3}$$

So that, $\nu(\xi,\tau)$ is an unknown function, the known functions $\nu_0(\tau), \nu_1(\tau)$, $g_0(\xi)$ *and* $g_1(\xi)$ are given continuous functions. Also, $q=max_{(\xi,\tau)\in\Omega}\{\delta(\xi,\tau)\}$ *and* $q\in Z^+$.

6.2 Function Approximation

Consider the basis function $\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)$ which is two variable function and can be expanded as:

$$\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)=G_{\tilde{m}}(\xi)\ \ell_{\tilde{n}}(\tau), \quad (\xi,\tau)\in\Omega=[0,1]\times[0,\infty), \tag{6.4}$$

where $\tilde{m}=0,1,...,\tilde{M}, \quad \tilde{n}=0,1,...,\tilde{N}, G_{\tilde{m}}(\xi)$ is the shifted Chebyshev polynomials defined on the interval $[0,1]$ and $\ell_n(\tau)$ is the shifted Laguerre polynomials defined on the interval $[0,\infty)$.
The shifted Chebyshev-Lagurre polynomials are orthogonals with respect to the weight function $\Xi(\xi,\tau)=exp(-\tau)$ [45, 68, 69]:

$$\int_0^\infty\int_0^1 \Xi(\xi,\tau)\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)\Phi_{ij}(\xi,\tau)\ d\xi d\tau=\frac{1}{2\tilde{m}+1}\delta_{\tilde{m}i}\delta_{\tilde{n}j}. \tag{6.5}$$

where $\delta_{\tilde{m}i}$ and $\delta_{\tilde{n}j}$ are the Kronecker functions.
Any function $\nu(\xi,\tau)\in L_2(\Omega)$ may be decomposed as:

$$\nu(\xi,\tau)=\sum_{\tilde{m}=0}^{\tilde{M}}\sum_{\tilde{n}=0}^{\tilde{N}}\nu_{\tilde{m}\tilde{n}}\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)\simeq G^T(\xi)V\ \ell(\tau), \tag{6.6}$$

where

$$\nu_{\tilde{m}\tilde{n}}=(2\tilde{m}+1)\int_0^\infty\int_0^1 \Xi(\xi,\tau)\nu(\xi,\tau)\Phi_{\tilde{m}\tilde{n}}(\xi,\tau)d\xi d\tau. \tag{6.7}$$

and

$$V=\begin{bmatrix}\nu_{00} & \nu_{01} & \nu_{02} & \cdots & \nu_{0\tilde{N}}\\ \nu_{10} & \nu_{11} & \nu_{12} & \cdots & \nu_{1\tilde{N}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \nu_{\tilde{M}0} & \nu_{\tilde{M}1} & \nu_{\tilde{M}2} & \cdots & \nu_{\tilde{M}\tilde{N}}\end{bmatrix},$$

$$G(\xi)=[G_0(\xi),\ G_1(\xi),\dots G_{\tilde{M}}(\xi)]^T and\ \ \ell(\tau)=[\ell_0(\tau),\ \ell_1(\tau),\dots \ell_{\tilde{N}}(\tau)]^T. \tag{6.8}$$

6.3 Pseudo-Operational Matrix of Integer Order Integral of the shifted Chebyshev and Laguerre Polynomials:

The purpose of this section is to find the OMs of the integer order quintessential of SCHPs and the SℓPs respectively using Taylor polynomials TPs [70–72], which is described as follows

$$T_k(\xi) = \xi^k, \qquad k = 0, 1, ..., M.$$

The (SCHPs) may be expressed by means of the TPs as:

$$G(\xi) = D_1\ T(\xi),$$

since

$$T(\xi) = [1, \xi, \xi^2, \ldots, \xi^M]^T, \qquad D_1 = [d^1_{\imath\jmath}]_{(M+1)\times(M+1)},$$

$$d^1_{\imath\jmath} = \begin{cases} \dfrac{(-1)^{\imath-\jmath}\ \jmath\ (\imath+\jmath-1)!}{(\imath-\jmath)!\ (2\jmath!)h^{\jmath}}, \imath \geq \jmath \\ 0, \qquad\qquad otherwise. \end{cases} \tag{6.9}$$

Then, by integrating $G(\xi)$, the pseudo-operational matrix of the (SCHPs) is obtained:

$$\int_0^{\xi} G(\rho)\ d\rho = \int_0^{\xi} D_1 T(\rho) d\rho = D_1 \int_0^{\xi} T(\rho) d\rho$$
$$= \xi D_1 \Lambda_1 T(\xi) = \xi D_1 \Lambda_1 D_1^{-1} G(\xi) = \xi \vartheta_1 G(\xi).$$

where $\vartheta_1 = D_1 \Lambda_1 D_1^{-1}$ is the pseudo-operational matrix of the integer order integral of the (SCHPs) and Λ_1 is defind by[42, 65, 73]:

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1/2 & 0 & ... & 0 \\ 0 & 0 & 1/3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(M+1) \end{bmatrix},$$

Similarly

$$\ell(\tau) = D_2 T(\tau), \tag{6.10}$$

where

$$d^2_{\imath\jmath} = \begin{cases} \dfrac{(-1)^{\imath}\ (\imath)!}{(\imath-\jmath)!\ (\jmath!)^2}, \imath \geq \jmath \\ 0, \qquad\qquad otherwise. \end{cases} \tag{6.11}$$

Also, with the aid of integrating $\ell(\tau)$,we gain the operational matrix of integer integration of the (SℓPs) as:

$$\int_0^\tau \ell(\rho)\ d\rho = \int_0^\tau D_2 T(\rho) d\rho = D_2 \int_0^\tau T(\rho) d\rho$$
$$= \tau D_2 \Lambda_2 T(\tau) = \tau D_2 \Lambda_2 D_2^{-1} \ell(\tau) = \tau \vartheta_2 \ell(\tau).$$

where $\vartheta_2 = D_2 \Lambda_2 D_2^{-1}$ is the pseudo operational matrix of the integer order integral of the shifted (SℓPs) and Λ_2 is given by:

$$\Lambda_2 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1/2 & 0 & ... & 0 \\ 0 & 0 & 1/3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(N+1) \end{bmatrix} ,$$

6.4 The Approach

This section is devoted for finding the numerical solution to the following VF-PDDEs in (6.1)-(6.3). For this problem assume that, the easiest order of spinoff with appreciate to ξ and τ is 2. Therefore, we obtain the following approximate functions as

$$\frac{\partial^4 \nu(\xi,\tau)}{\partial \xi^2 \partial \tau^2} \simeq G^T(\xi) U \ell(\tau), \tag{6.12}$$

where the unknown matrix U is defined as follows:

$$U = \begin{bmatrix} u_{00} & u_{01} & u_{02} & ... & u_{0N} \\ u_{10} & u_{11} & u_{12} & ... & u_{1N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{M0} & u_{M1} & u_{M2} & ... & u_{MN} \end{bmatrix} .$$

By integrating of the above equation (6.12) with respect to τ and using the initial condition (6.3), we have:

$$\frac{\partial^3 \nu(\xi,\tau)}{\partial \xi^2 \partial \tau} \simeq \tau G^T(\xi) U \vartheta_2 \ell(\tau) + \acute{g_1}(\xi). \tag{6.13}$$

Integrating (6.13) with respect to τ, yields:

$$\frac{\partial^2 \nu(\xi,\tau)}{\partial \xi^2} \simeq \tau^2 G^T(\xi) U \vartheta_2 \hat{\vartheta}_2 \ell(\tau) + \tau \acute{g_1}(\xi) + \acute{g_0}(\xi), \tag{6.14}$$

where

$$\begin{aligned}\int_0^{\tau}\rho L(\rho)d\rho &= \int_0^{\tau}\rho D_2 T(\rho)d\rho = D_2\int_0^{\tau}\rho T(\rho)d\rho \\ &= \tau^2 D_2\hat{\Lambda}_2 T(\tau) = \tau^2 D_2\hat{\Lambda}_2 D_2^{-1}\ell(\tau) = \tau^2\hat{\vartheta}_2\ell(\tau),\end{aligned} \tag{6.15}$$

and

$$\hat{\Lambda}_2 = \begin{bmatrix} 1/2 & 0 & 0 & ... & 0 \\ 0 & 1/3 & 0 & ... & 0 \\ 0 & 0 & 1/4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(N+2) \end{bmatrix}.$$

Now, by integrating (6.14) with respect to ξ, we get

$$\begin{aligned}\frac{\partial\nu(\xi,\tau)}{\partial\xi} &\simeq \xi\ \tau^2 G^T(\xi)\vartheta_1^T U\vartheta_2\hat{\vartheta}_2\ell(\tau) + \tau(\acute{g}_1(\xi) - \acute{g}_0(0)) + (\acute{g}_0(\xi) - \acute{g}_0(0)) \\ &+ \frac{\partial\nu(0,\tau)}{\partial\xi},\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}\nu(\xi,\tau) &\simeq \xi^2\ \tau^2 G^T(\xi)\hat{\vartheta}_1^T\ \vartheta_1^T U\ \vartheta_2\hat{\vartheta}_2\ell(\tau) + \tau(g_1(\xi) - g_1(0) - \xi\ \acute{g}_1(0)) \\ &+ (g_0(\xi) - g_0(0) - \xi\ \acute{g}_0(0)) + \xi\ \frac{\partial\nu(\xi,0)}{\partial\xi} + \nu_0(\tau),\end{aligned} \tag{6.17}$$

where

$$\begin{aligned}\int_0^{\xi}\rho P(\rho)d\rho &= \int_0^{\xi}\rho D_1 T(\rho)d\rho = D_1\int_0^{\xi}\rho T(\rho)d\rho \\ &= \xi^2 D_1\hat{\Lambda}_1 T(\xi) = \xi^2 D_1\hat{\Lambda}_1 D_1^{-1}G(\xi) = \xi^2\hat{\vartheta}_1 G(\xi),\end{aligned} \tag{6.18}$$

and

$$\hat{\Lambda}_1 = \begin{bmatrix} 1/2 & 0 & 0 & ... & 0 \\ 0 & 1/3 & 0 & ... & 0 \\ 0 & 0 & 1/4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1/(M+2) \end{bmatrix}.$$

Integrating (6.12) w.r.t. ξ and by the aid of the conditions (6.2) and (6.3), yields:

$$\frac{\partial^3\nu(\xi,\tau)}{\partial\xi\partial\tau^2} \simeq \xi G^T(\tau)\vartheta_1^T U\ell(\tau) + \frac{\partial^3\nu(0,\tau)}{\partial x\partial\tau^2}. \tag{6.19}$$

$$\frac{\partial^2\nu(\xi,\tau)}{\partial\tau^2} \simeq \xi^2 G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ell(\tau) + \xi\ \frac{\partial^3\nu(0,\tau)}{\partial\xi\partial\tau^2} + \acute{\nu}_0(\tau). \tag{6.20}$$

It is remarkable that $\frac{\partial^3\nu(0,\tau)}{\partial\xi\partial\tau^2}$ is unknown function, by integrating (6.19) from 0 to 1 with respect to ξ, we get:

$$\frac{\partial^3\nu(0,\tau)}{\partial\xi\partial\tau^2} \simeq \acute{\acute{\nu_1}}(\tau) - \acute{\acute{\nu_0}}(\tau) - S^T D_1^T \vartheta_1^T U\ell(\tau),$$

where

$$\int_0^1 \xi G^T(\rho)d\xi = \int_0^1 \xi T(\xi)D_1^T d\xi = S^T D_1^T,$$

and

$$S = [\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{M+2}]^T.$$

Then

$$\frac{\partial^2\nu(\xi,\tau)}{\partial\tau^2} \simeq \xi^2 G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ell(\tau) + x[\acute{\acute{\nu_1}}(\tau) - \acute{\acute{\nu_0}}(\tau) - S^T D_1^T\vartheta_1^T U\ell(\tau)] + \acute{\acute{\nu_0}}(\tau). \tag{6.21}$$

By integrating (6.21)for τ, we acquire to

$$\frac{\partial\nu(\xi,\tau)}{\partial\tau} \simeq \xi^2\tau G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\vartheta_2\ell(\tau) + \xi[\acute{\nu}_1(\tau) - \acute{\nu}_0(\tau) - \tau S^T D_1^T\vartheta_1^T U\vartheta_2\ell(\tau)] + \acute{\nu}_0(\tau) + g_1(\tau) \tag{6.22}$$

6.4.1 The Operational Matrix of the delay term:

In this subsection, the delay term $\nu(\xi,\tau-\kappa)$ will be approximated by using the operational matrix of the Laguerre polynomials as follows:
consider[42]:

$$\nu(\xi,\tau-\kappa) = G^T(\xi)U\ell(\tau-\kappa), \tag{6.23}$$

where

$$\ell(\tau-\kappa) = HP^T(\tau-\kappa), \tag{6.24}$$

and

$$H = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & ... & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & ... & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & ... & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}.$$

To get $P(\tau - k)$ by means of $P(\tau)$, we must employ the next relation:

$$P(\tau) = [1, \tau, \tau^2, ..., \tau^N], \quad P(\tau - \kappa) = [1, \tau - \kappa, (\tau - k)^2, ...(\tau - k)^N].$$

$$P(\tau - k) = P(\tau) B_{-k}^T, \tag{6.25}$$

where

$$B_{-\kappa}^T = \begin{bmatrix} \binom{0}{0}(-\kappa)^0 & \binom{1}{0}(-\kappa)^1 & \binom{2}{0}(-\kappa)^2 & \cdots & \binom{N}{0}(-\kappa)^N \\ 0 & \binom{1}{1}(-\kappa)^0 & \binom{2}{1}(-\kappa)^1 & \cdots & \binom{N}{1}(-\kappa)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\kappa)^0 & \cdots & \binom{N}{2}(-\kappa)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{2}(-\kappa)^0 \end{bmatrix}.$$

By using equs. (6.23) - (6.25), we have:

$$\nu(\xi, \tau - \kappa) = G^T(\xi) U B_{-\kappa}^T H^T \ell(\tau). \tag{6.26}$$

6.4.2 Computation of (VFD) of $\nu(\xi,\tau)$:

Here, we expand $D_\tau^{\delta(\xi,\tau)}, 0 < \delta(\xi,\tau) \leq 1$ in terms of the (SℓPs), by using equ. (6.22), we get:

$$\begin{aligned} D_\tau^{\delta(\xi,\tau)}\nu(\xi,\tau) &= I_\tau^{1-\delta(\xi,\tau)}(\frac{\partial\nu(\xi,\tau)}{\partial\tau}) \\ &\simeq \xi^2\tau^{2-\delta(\xi,\tau)}G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ \vartheta_2\ \hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau) + \xi I_\tau^{1-\delta(\xi,\tau)}(\acute{\nu}_1(\tau) - \acute{\nu}_0(\tau)) \\ &- \xi\tau^{2-\delta(\xi,\tau)}S^T D_1^T\vartheta_1^T U\vartheta_2\ \hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau) + \frac{\Gamma(1)}{\Gamma(2-\delta(\xi,\tau)}\tau^{1-\delta(\xi,\tau)}g_1(\xi) \\ &+ I_\tau^{1-\delta(\xi,\tau)}\acute{\nu}_0(\tau). \end{aligned} \tag{6.27}$$

So that

$$I_\tau^{1-\delta(\xi,\tau)}(\tau\ell(\tau)) \simeq \tau^{2-\delta(\xi,\tau)}\hat{\Theta}_N^{1-\delta(\xi,\tau)}\ell(\tau),$$

since

$$\hat{\Theta}_N^{1-\delta(\xi,\tau)} = D_2\hat{\vartheta}_N^{1-\delta(\xi,\tau)}D_2^{-1}.$$

Also, for $1 < \delta(\xi,\tau) \leq 2$,

$$\begin{aligned} D_\tau^{\delta(\xi,\tau)}\nu(\xi,\tau) &= I_\tau^{2-\delta(\xi,\tau)}(\frac{\partial^2\nu(\xi,\tau)}{\partial\tau^2}) \\ &\simeq \xi^2\tau^{2-\delta(\xi,\tau)}G^T(\xi)\hat{\vartheta}_1^T\vartheta_1^T U\ \vartheta_2\ \hat{\Theta}_N^{2-\delta(\xi,\tau)}\ell(\tau) + \xi I_\tau^{2-\delta(\xi,\tau)}(\acute{\acute{\nu}}_1(\tau) - \acute{\acute{\nu}}_0(\tau)) \\ &- \xi\tau^{2-\delta(\xi,\tau)}S^T D_1^T\vartheta_1^T U\vartheta_2\ \hat{\Theta}_N^{2-\delta(\xi,\tau)}\ell(\tau) + I_\tau^{2-\delta(\xi,\tau)}\acute{\acute{\nu}}_0(\tau). \end{aligned} \tag{6.28}$$

Substituting the approximations (6.15), (6.14)and (6.28) into equ. (6.1) and the nodal points of Newton-Cotes,[74], then we get an algebraic system of equations and by using the Newtons iterative method. We get the unknown matrix U. Substituting U into eq. (6.17), we attain the approximate solution of the problem (6.1)-(6.3).

6.5 Convergence analysis of approximate solution

The next theorem tell us that the Shifted Chebyshev-Lagurre operational matrices can approximating an arbitrary continuous function [75]

6.5.1 Maximum error

We demonstrate uniform convergence of the Chebyshev-Laguerre expansion of the continuous function $\nu(\xi,\tau)$. Prior to that, however, we offer the upper bound for its error as follows:
Let $Q_{N,M}$ be a collection of all polynomials with maximum degrees of N for ξ and M for τ. Therefore, there exists a unique $q_{N,M} \in Q_{N,M}$ such that for $\nu \in C(\Omega)$.

$$\|\nu(\xi,\tau) - \nu_{N,M}(\xi,\tau)\| \leq \|\nu(\xi,\tau) - q_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)}. \tag{6.29}$$

and we define

$$L^2_\omega(\Omega) = \{\Phi : \Phi \textit{ is measurable on } \Omega \textit{ and } \|\Phi\| < \infty\},$$

Assume $w(x)$ is a square integrable function with respect to the Chebyshev weight function $\chi_h(x)$ in $[0,h]$. Then it can be expressed by means of $T_j(x)$ as

$$w(x) = \sum_{j=0}^{\infty} c_j T_j(x).$$

The coefficients c_j are obtained from

$$c_j = \int_0^h \chi_h(x) w(x) T_j(x) dx, \quad j = 0, 1, \dots \ .$$

Definition 6.1.

If $\nu(\xi,\tau)$ is a function with two variables and it's continuous at the point (ξ_0,τ_0) we have all its partial derivatives are also continuous at that point, then by Taylor series of $\nu(\xi,\tau)$ about the point (ξ_0,τ_0) it is calculated as:

$$\nu(\xi,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!m!}\frac{\partial^m}{\partial\tau^m}(\frac{\partial^n \nu}{\partial\xi^n})|_{(\xi_0,\tau_0)}(\xi-\xi_0)^n(\tau-\tau_0)^m$$

Also, we have

$$\nu(\xi,\tau) = \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{1}{n!m!}\frac{\partial^m}{\partial\tau^m}(\frac{\partial^n \nu}{\partial\xi^n})|_{(\xi_0,\tau_0)}(\xi-\xi_0)^n(\tau-\tau_0)^m + R_{NM}(\xi,\tau). \tag{6.30}$$

As well as, where all partial derivatives of ν of order $N+M+2$ belong to $L^2_\omega(\Omega)$, then

$$|R_{NM}(\xi,\tau)| \leq \frac{(\xi-\xi_0)^{N+1}(\tau-\tau_0)^{M+1}}{(N+1)!(M+1)!} \times sup_{(\xi,\tau)\in(\Omega)}|\frac{\partial^{N+M+2}\nu(\xi,\tau)}{\partial\xi^{N+1}\partial\tau^{M+1}}| \tag{6.31}$$

Theorem 6.1.

Assume that the Chebyshev-Laguerre functions in are used to expand the real sufficiently smooth function ν, as

$$\nu(\xi,\tau) = \sum_{n=0}^{N}\sum_{m=0}^{M} \widetilde{\nu}_{nm}(\xi,\tau)\Psi_{nm}(\xi,\tau) = \widetilde{V}^T\Psi_{nm}(\xi,\tau),$$

where

$$\Psi_{nm}(\xi,\tau) = [\Psi_{00}(\xi,\tau), \Psi_{01}(\xi,\tau), ..., \Psi_{0M}(\xi,\tau), ..., \Psi_{N0}(\xi,\tau), \Psi_{N2}(\xi,\tau), ..., \Psi_{NM}(\xi,\tau)]^T,$$

$$\widetilde{V} = [\widetilde{\nu}_{00}, \widetilde{\nu}_{01}, ..., \widetilde{\nu}_{0M}, ..., \widetilde{\nu}_{M0}, \widetilde{\nu}_{M1}, ..., \widetilde{\nu}_{NM}]$$

if the magnitude of the bounded on the right side of (6.31) by

$$C_{NM} = \ sup_{(\xi,\tau)\in(\Omega)}|\frac{\partial^{N+M+2}\nu(\xi,\tau)}{\partial\xi^{N+1}\partial\tau^{M+1}}|$$

The upper bound of error can be calculated as

$$\|\nu(\xi,\tau) - \nu_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)} \leq \frac{C_{NM}\sqrt{(2M+2)!}}{(N+1)!(M+1)!\sqrt{(2N+3)}}. \tag{6.32}$$

Assume that

$$\widetilde{\nu}_{NM}(\xi,\tau) = \widetilde{V}^T\Psi_{NM}(\xi,\tau);$$

be the rough resolution obtained using the technique suggested in Section 6.4, where

$$\widetilde{V} = [\widetilde{\nu}_{00}, \widetilde{\nu}_{01}, ..., \widetilde{\nu}_{0M}, ..., \widetilde{\nu}_{M0}, \widetilde{\nu}_{M1}, ..., \widetilde{\nu}_{NM}].$$

Then

$$\|\nu(\xi,\tau) - \nu_{N,M}(\xi,\tau)\|_{L^2_\omega(\Omega)} \leq \frac{C_{NM}}{(N+1)!(M+1)!} \times \sqrt{\frac{(2M+2)!}{(2N+3)}} + \Theta_{NM}\|V - \widetilde{V}\|_2, \tag{6.33}$$

where

$$\Theta_{NM} = \sum_{n=0}^{N}\sqrt{\frac{M+1}{2n+1}},$$

and the norm $\|.\|_2$ is the standard Euclidean vector norm.

Proof: see the proof of theorem 3.1

6.6 Numerical Examples

To demonstrate the ability of the proposed method for solving (VFDDEs), two tested examples given:

Example 6.1.

Consider the (VFDDEs) (6.1) with $\eta = 1, \kappa = 0.1$ and subject to:

$$\nu(0,\tau) = 0, \quad \nu(1,\tau) = 0, \quad \tau \in [0,\infty), \tag{6.34}$$

$$\nu(\xi,0) = 10\xi^2\ (1-\xi)^2, \qquad \frac{\partial \nu(\xi,0)}{\partial \tau} = 0, \tag{6.35}$$

where

$$f(\nu(\xi,\tau),\nu(\xi,\tau-\kappa)) = 10\xi^2(1-\xi)^2\frac{\tau^{2-\delta(\xi,\tau)}}{\Gamma(3-\delta(\xi,\tau))} - 20(6\xi^2-6\xi+1)(\tau^2+1) - 10(\tau-0.1+1)^2\xi^2(1-\xi)^2.$$

This problem has an exact solution $\nu(\xi,\tau) = 10\xi^2(1-\xi)^2(\tau^2+1)$ and

$$\delta(\xi,\tau) = 1.8 - 0.005\ cos(\xi\tau)\ sin(x).$$

Figs.6.1 and 6.2 represent the (AE) of example 6.1 for M=N=8 and h=10 and distinct values of $\delta(\xi,\tau)$. Also, Figs.6.3 and 6.4 represent the (AE) in 3D.

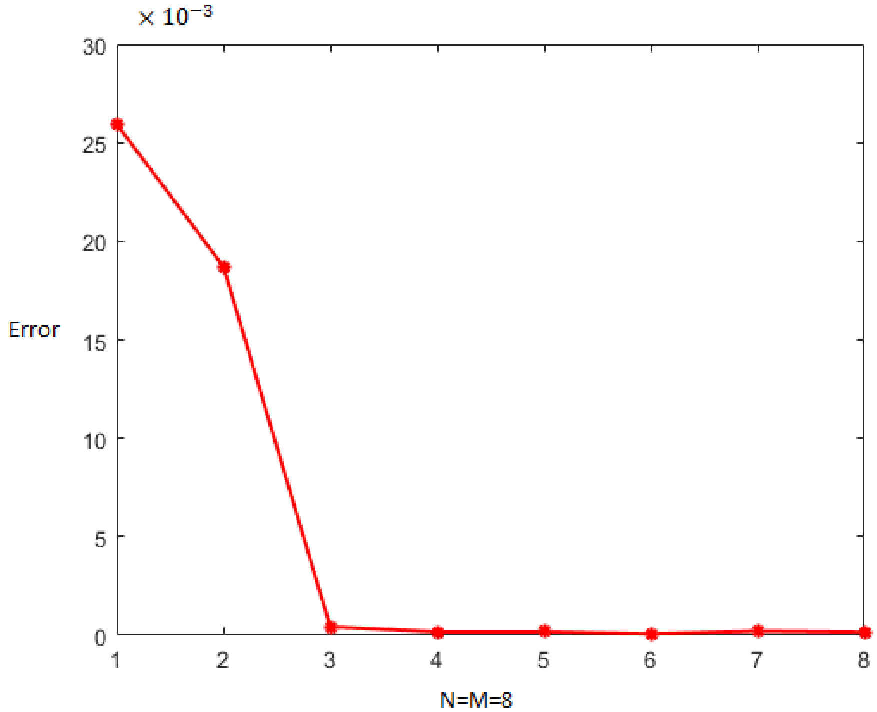

Figure 6.1: Absolute error of example 6.1 at $\delta(\xi,\tau) = \frac{9}{5} - 0.005\ cos(\tau\xi)\ sin(\xi), \xi = \tau = 0.7$

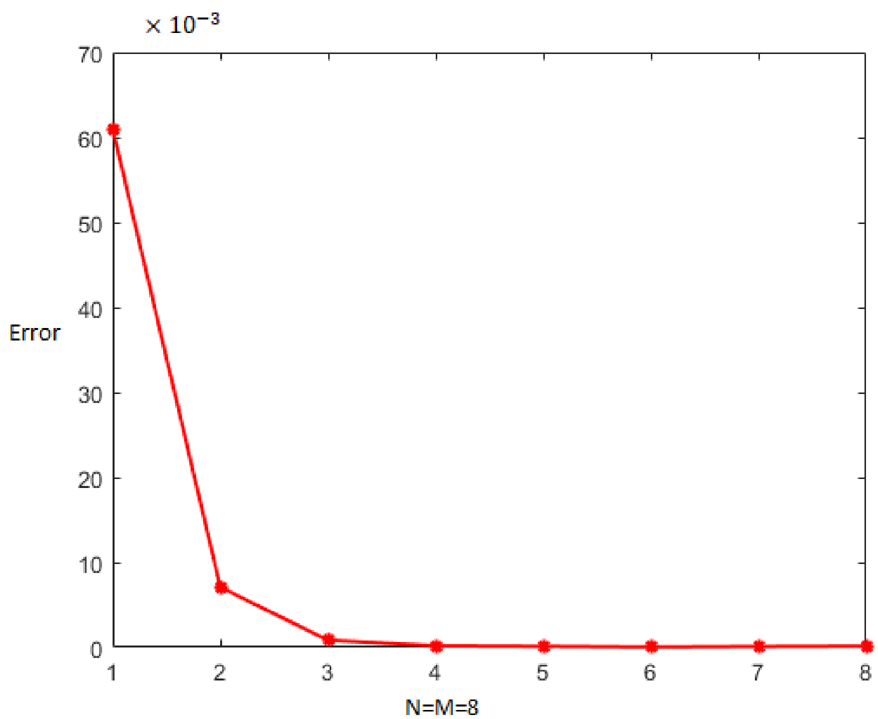

Figure 6.2: Absolute error of example 6.1 at $\delta = 1.7 + e^{-\xi\tau}, \xi = \tau = 0.8$

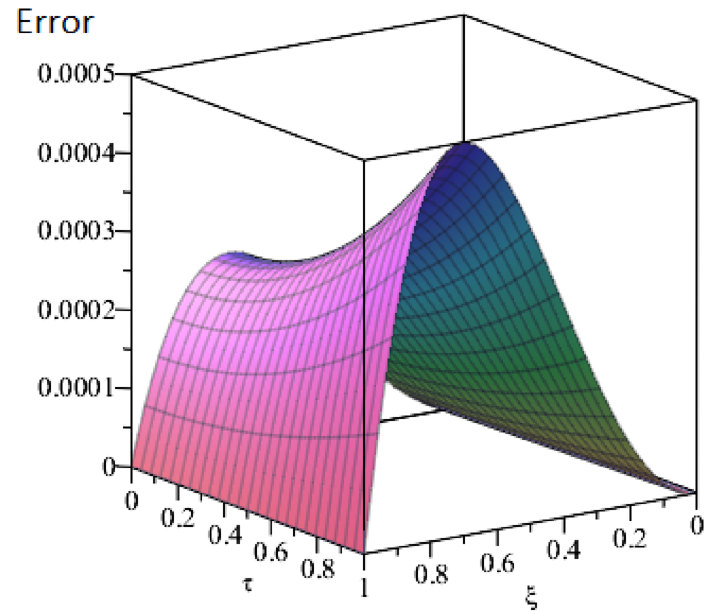

Figure 6.3: Absolute error of example 6.1 at $\delta(\xi,\tau) = 1.8 - 0.005cos(\xi\tau)sin(\tau)$

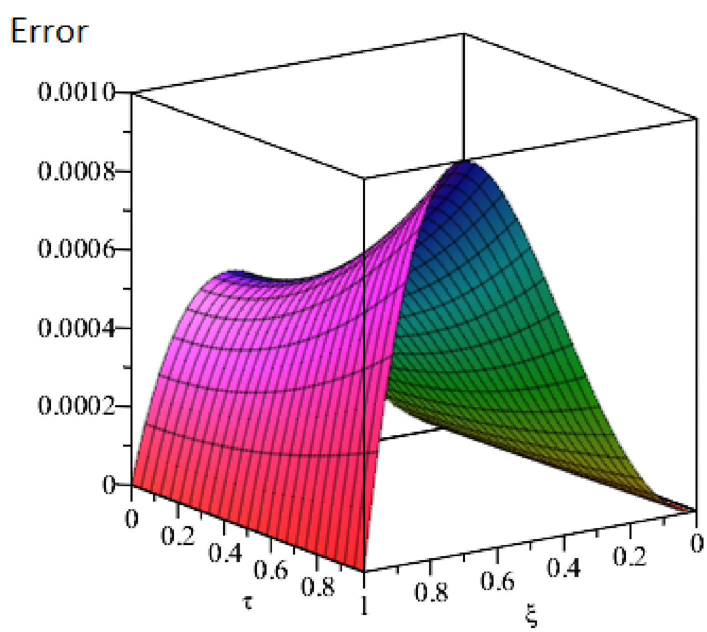

Figure 6.4: Absolute error of example 6.1 at $\delta(\xi,\tau) = 1.5 + 0.4e^{-\xi\tau}$,

Example 6.2.

Consider the VFDDEs (6.1) with $\eta = 1, \quad \kappa = 0.2$ and subject to:

$$\nu(0,\tau) = 0, \qquad \nu(1,\tau) = 0, \qquad \tau \in [0,\infty), \tag{6.36}$$

$$\nu(\xi,0) = \frac{\partial \nu(\xi,0)}{\partial \tau} = 5\xi\ (1-\xi), \tag{6.37}$$

where

$$f(\nu(\xi,\tau),\nu(\xi,\tau-\kappa)) = 5\xi(1-\xi)\frac{\tau^{1-\delta(\xi,\tau)}}{\Gamma(2-\delta(\xi,\tau))} - 10\tau + 5\xi(1-\xi)(\tau-0.2+1).$$

This problem has an exact solution $\nu(\xi,\tau) = 5\xi(1-\xi)(\tau+1)$ and

$$\delta(\xi,\tau) = 2 - 0.2\ cos(\tau)\ sin(\xi).$$

Figure.6.5 and 6.6 represent (AE) of example 6.2 for M=N=8 , h=10 and distinct values of $\delta(\xi,\tau)$. Also, Figs.6.7 and 6.8 represent the (AE) in 3D.

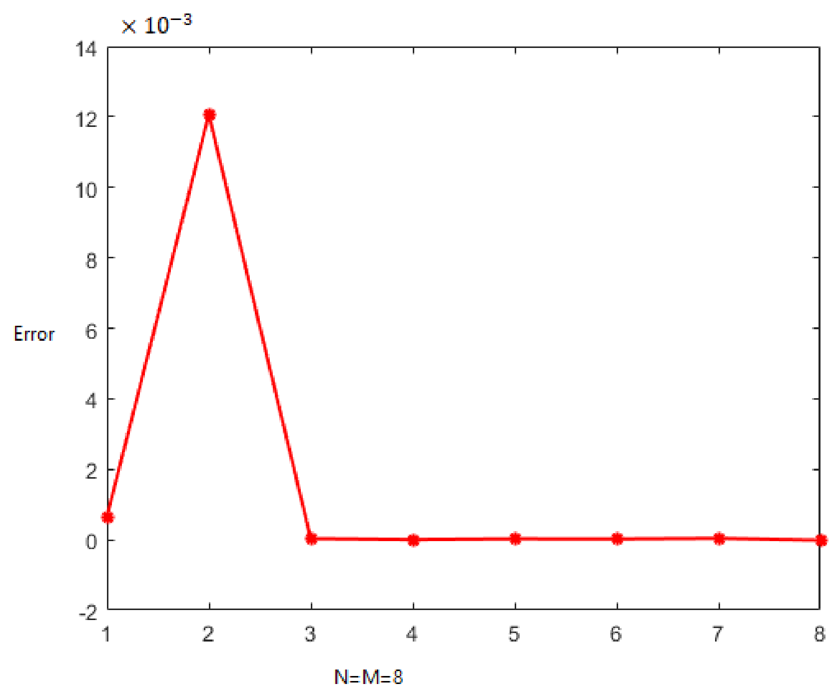

Figure 6.5: Absolute error of example 6.2 at $\delta(\xi,\tau) = 2 - cos(\xi)sin(\tau), \xi = \tau = 0.8$

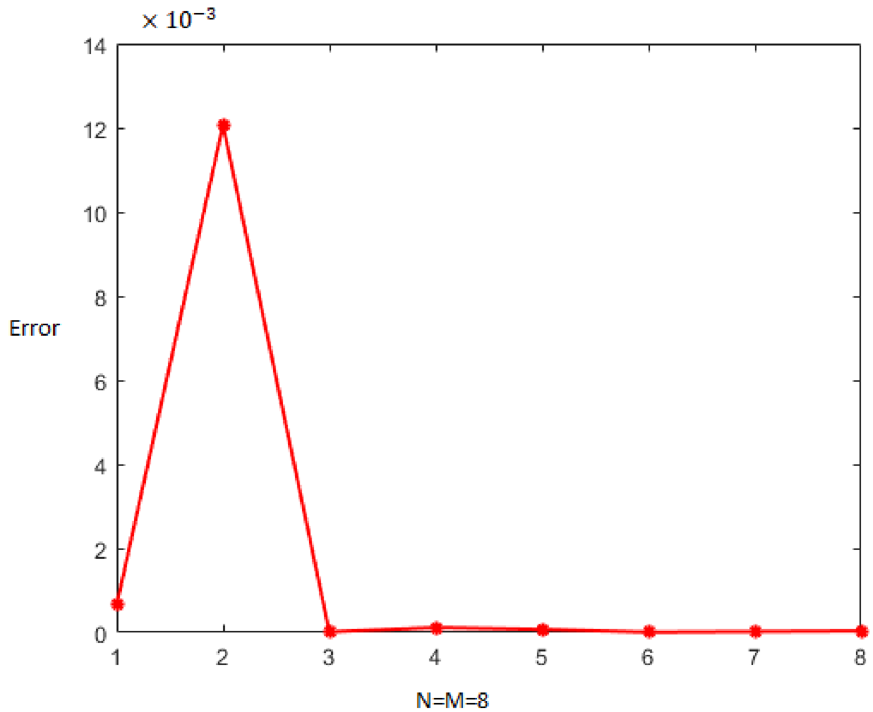

Figure 6.6: Absolute error of example 6.2 at $\delta(\xi,\tau) = 1 + \frac{1}{2}sin(\xi\tau)e^{-\tau}, \xi = \tau = 0.7$

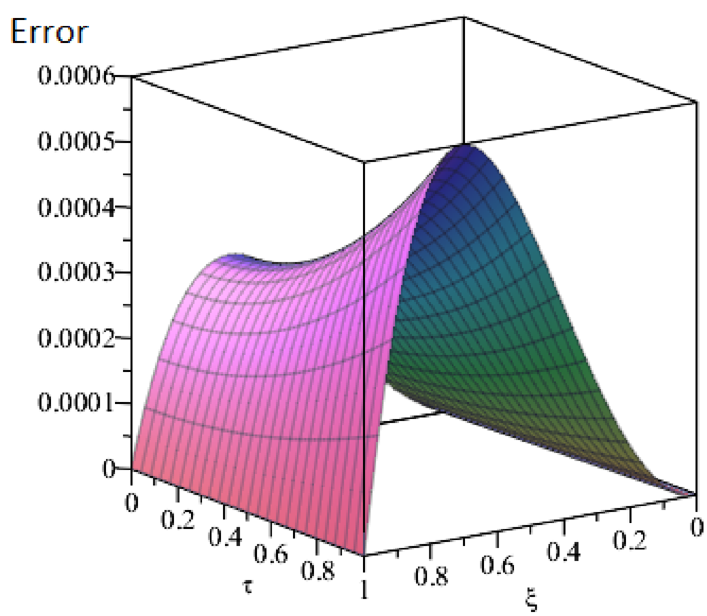

Figure 6.7: Absolute error of example 6.2 at $\delta(\xi,\tau) = 2 - cos(\xi)sin(\tau)$

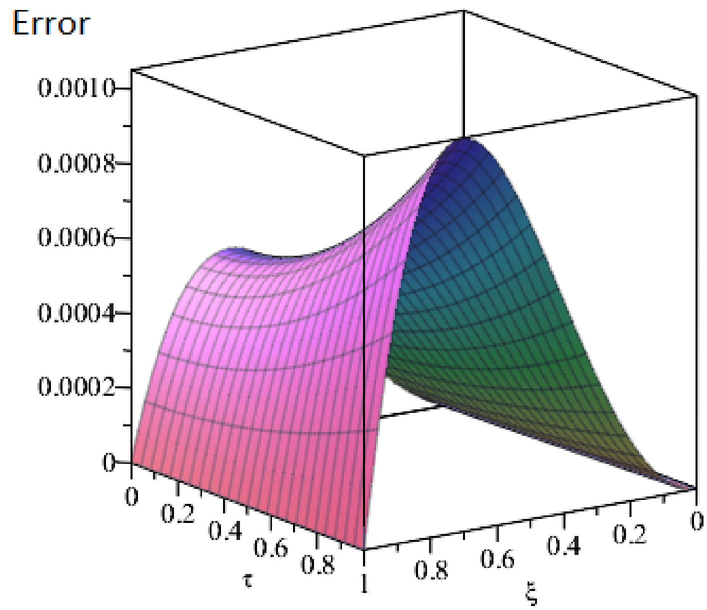

Figure 6.8: Absolute error of example 6.2 at $\delta(\xi,\tau) = 1 + 0.5sin(\tau)e^{-\tau}$

Chapter 7

Conclusions and Future Works

7.1 Conclusions

In this thesis, we propose the collocation method and the (OMs) of shifted Legendre-Laguerre polynomials to approximate the solutions of (VFDDEs). The proposed method transform the(VFDDEs)to system of algebraic equations using the nodal points of Newton-Cots. By solving the algebraic system using Newtons iterative methods, numerical solutions are obtained. The numerical results approved that the proposed method is accurate and has very high accuracy as M and N increased.

The HPM is used in this study to approximate the solutions of VFPDEs with proportional delays. The variable order fractional derivative was approximated in terms of the standard derivative. Comparatively speaking, the proposed method requires a lot less computational work than other numerical methods. The obtained results, compared with exact solution, show us that this method is remarkable very effective, very simple, and very fast in convergence for handling VFPDEs with proportional delays. In this thesis Maple 2016 was used for all calculations.

In order to estimate the solution of (VOFPDEs), we propose the collocation approach in this study along with the operational matrices of shifted Chebyshev polynomials. When we use the suggested method, a set of algebraic nonlinear equations will appear. Newton's iteration approach was used to solve the resulting algebraic system in order to arrive at the desired solutions. The evaluated instances confirmed since the suggested approach is precise and extremely accurate.

7.2 Future Works

In this thesis, our main a general study for solving a model of variable-order fractional delay equations the following is addressed:

1. Extending the proposed numerical techniques given in this thesis to multi-dimensional problems and their stabilization.
2. Developing and testing new numerical methods for solving variable order fractional partial differential equations with delay.
3. Studying the properties of variable order fractional partial differential equations, such as positivity, monotonicity, and maximum principles, and adapting the numerical methods to these properties.
4. Developing adaptive numerical methods that can change the order of the fractional derivatives based on the solution's behavior.
5. Investigating the numerical solutions of other types of fractional partial differential equations, such as stochastic or fuzzy fractional partial differential equations.
6. Developing analytical methods to obtain the exact solutions of variable order fractional partial differential equations as a comparison to numerical methods.
7. Developing analytical methods to obtain the exact solutions of variable order fractional partial differential equations in chapter 3 and 6 by using Jacobi-Laguerre polynomials.

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